FRACTIONAL AND DYNAMICAL INTEGRAL INEQUALITIES WITH APPLICATIONS

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LAHORE – PAKISTAN

2019

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This dissertation is submitted to the Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan, for the partial fulfillment of the requirements for the award of the degree of

Doctor of Philosophy

In

MATHEMATICS

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2014-Ph.D-Math-01

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2019

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I **Sobia Rafeeq** hereby state that my PhD thesis titled:

FRACTIONAL AND DYNAMICAL INTEGRAL INEQUALITIES WITH APPLICATIONS

is my own work and has not been submitted previously by me for taking any degree from this University

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or anywhere else in the country/world.

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بسم الله الرحمن الرحيم

In the Creation of Heaven and Earth,
In the Alternation of Night and Day,
In the Ships that Sail and Benefit The Man,
In the Clouds, In the Rain,
There are Signs for
Those
Who
Think, Believe &
Understand.

Al-Quran

DEDICATION

I would like to dedicate this thesis to my mot

Acknowledgement

All gratitude and praise be to ALMIGHTY ALLAH alone, the most gracious, most merciful. It was his blessing which enabled me to complete the thesis successsfully. I offer my humble and sincerest words of thanks to HOLY PROPHET(P.BU.H), who is a torch of knowledge and guidance for mankind and all generations to come.

This study work would not have been possible without the guidance and the help of several individuals, who in one way or another, contributed and extended their valuable assistance in the preparation and completion of this study. First my utmost gratitude to **Dr. Sabir Hussain** "Associate Professor, Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan", whose kind supervision and encouragement, I will never forget. His untiring effort, constructive criticism, commitment, guidance and support helped me greatly in the understanding and writing of the dissertation. He remained a source of inspiration for me throughout my study and research work.

I would like to sincerely thanks to **Prof. Dr. Mohammad Ozair Ahmed**, "Head of the department of Mathematics, The University of Lahore" and **Dr. Asma Rashid Butt**, "Associate Professor, Department of Mathematics, University of Engineering and Technology, Lahore", for their encouraging attitude and patent advice during the course of conducting the research work and compilation of this document.

I would also like to thank **Prof. Dr. Muhammad Mushtaq**, "Chairman, Department of Mathematics, University of Engineering and Technology, Lahore for his invaluable guidance, help and encouragement, without which i feel difficult to get this arduous job completed. I acknowledge with thanks the cooperation of all the staff members of Mathematics department for their valuable suggestions and appreciable guidance. I also deem my gratitude to the non-teaching and technical staff of the department, who always kept their assistance, stands by whenever it was required.

Lastly, I am highly gratified to my beloved Mother and brother, who are my heartiest mentors, for their continuous encouragement to complete this work. Without their payers, sacrifices and encouragement, the present work would have been merely a distant dream.

Sobia Rafeeq

Abstract

Fractional calculus, the study of of integration and differentiation of fractional order, has recently been extended to include its discrete analogues of fractional difference calculus and fractional quantum calculus. Due to this, there was a question whether there exist a single theory pertaining the above theory. Answer to this task was proposed by great scientist Stephen Hilger (1988), P. A. Williams and Bastos (2012), Jiang Zhu et al. (2013). This field is diverse, having a lot of applications in different fields of sciences such as: differential equations, probability theory, Mathematical and Economical models, optimization theory, signal processing, chaotic dynamics, atomic Bose-Einstein condensation, theory of inequalities etc.

Inequalities play a significant role in many branches of Sciences as well as to discuss the abstract analysis of the solutions of differential, difference equations and Cauchy type problems. Among others inequalities, Gronwall-Bellman type integral inequalities have a significant part in this direction. We propose $\alpha-$Delta integral, $\Delta-$multi-time scale integral, $It\hat{o}-$Isometry, generalized fractional dynamical Gronwall-Bellman type integral inequalities to analyze some qualitative and quantitative properties of solutions of integro-differential equations, cauchy type problems, nonlinear fractional stochastic differential equation and fractional $\Delta-$stochastic differential equation.

CONTENTS

CHAPTER 1

Preliminary discussion

1.1 Introduction

Fractional calculus is a generalization of ordinary calculus in which we study derivatives and integrals of fractional order. The study of fractional calculus started in the 1695 in a correspondence between the pioneers of calculus, L' Hospital and Leibnitz. The beauty of fractional calculus is that it translates the reality of nature in a better way period. Fractional calculus provides more accurate results of the physical systems than ordinary calculus do. Fractional derivatives is an excellent instrument for the description of long-term memory and chaotic behavior of various materials and processes. These effects were neglected in classical integer-order models, this is the main advantage of fractional calculus.

Fractional calculus has become very useful over the last forty years due to its demonstrated applications in almost all the applied sciences. We now see applications in acoustic wave propagation in inhomogeneous porous material fluid flow, dynamics of earth quakes, bioengineering, medicine, economics, statistics, astrophysics, chemical engineering, nonlinear control, control of electronic power, and neural networks, among others [2]. By now almost all fields of research in science and engineering use fractional calculus due to the necessity of dealing with fractional phenomena and structures. So, this field is keeping a lot of people active and interested.

For a long time, it was considered that fractional derivatives and integrals have no evident geometrical meaning due to their nonlocal behavior. In 2002, I. Podlubny shown that geometric interpretation of Riemann-Liouville fractional integration is "Shadow on the walls" [21]. In 2016, V. E. Tarasov has given the geometrical interpretation of the Riemann-Liouville fractional derivative by using the concept of osculation and linked it with the geometrical interpretation of ordinary derivative. The concept of osculation is a generalization of the concept of tangency and the tangent line. Two functions have a *contact of order* $\mathfrak{n}$ at a point $\mathfrak{x}_0$ if they have the same value and $\mathfrak{n}$ equal derivatives at this point. In the geometry of curves and surfaces, the geometric objects which have contact of order $\mathfrak{n}$ at a point is called the *osculating objects*. An osculating curve $\mathfrak{y} = \mathfrak{f}(\mathfrak{x})$ from a given family of curves is a curve that has the highest possible order of contact with a given curve $\mathfrak{y} = \mathfrak{g}(\mathfrak{x})$ at a given point $\mathfrak{x}_0$. For example, a tangent line is an osculating curve from the family of straight lines, and it has first order contact with the given curve $\mathfrak{y} = \mathfrak{f}(\mathfrak{x})$. An osculating circle is an osculating curve from the family of circles and it has second order contact.

The notion of $\mathfrak{n}-$order contact at a point allows us to define an equivalence relation. Contact of order $\mathfrak{n}$ at $\mathfrak{x}$ can be considered as an equivalence relation of the space of functions. The equivalence class of smooth functions with respect to the equivalence relation of contact of order $\mathfrak{n}$ at $\mathfrak{x}$ is called the $\mathfrak{n}-$*jet of function* at $\mathfrak{x}$. The geometric interpretation of integer-order derivatives is *finite-order jet of functions* while geometric interpretation of fractional-order derivatives is *infinite-order jet of functions* [25].

The theory of time scales is a growing new area of research both in theories and applications. Time scale is applicable to any field in which the dynamic processes are discrete and continuous. For example, it can model plant population of one particular species which grows exponentially during the months of April until September, all plants die at the beginning of October, but the seeds remain in the ground and start growing again at the beginning of April.

This observations give birth to a general theory, time scales. The motivation for such general theory is rooted in the fact that there is a disconnect between discrete and continuous methods. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be different from continuous counterparts. Unification of these two types of dynamic equations in a general theory will help explain these similarities and discrepancies. In addition, times scales can be used to study problems that cannot be approached with differential and difference equations. So, unification and extension are the two main features of the time scales calculus. The time scales calculus has a tremendous potential for applications in the field of economy, physics, engineering, medical sciences and entomology.

1.2 Time Scales Essentials

Definition 1.2.1 [5] *A time scale is an arbitrary nonempty closed subset of the real numbers. It is usually denoted by* $\mathbb{T}$.

Definition 1.2.2 *The forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *for* $\mathfrak{s} \in \mathbb{T}$ *is defined as*

$$\sigma(\mathfrak{s}) := \inf\{r \in \mathbb{T} : r > \mathfrak{s}\}.$$

The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ *for* $\mathfrak{s} \in \mathbb{T}$ *is defined as*

$$\rho(\mathfrak{s}) := \sup\{r \in \mathbb{T} : r < \mathfrak{s}\}.$$

Example 1.2.1 *Let* $\mathbb{T} = \{\frac{1}{\omega} : \omega \in \mathbf{N}\} \cup \{0\}$. *By using the definition of forward jump operator*

$$\begin{aligned} \sigma\left(\frac{1}{\mathfrak{s}}\right) &= \inf\left\{\frac{1}{r} \in \mathbb{T} : \frac{1}{r} > \frac{1}{\mathfrak{s}}\right\} \\ &= \inf\left\{\frac{1}{\mathfrak{s}-1}, \frac{1}{\mathfrak{s}-2}, \frac{1}{\mathfrak{s}-3}, \cdots\right\} \\ \sigma\left(\frac{1}{\mathfrak{s}}\right) &= \frac{1}{\mathfrak{s}-1},\ \mathfrak{s} \neq 1. \\ \Rightarrow \sigma(\omega) &= \frac{\omega}{1-\omega},\ \omega \neq 1. \end{aligned}$$

When $\omega = 1$,

$$\begin{aligned} \sigma(1) &= \inf\{r \in \mathbb{T} : r > 1\} \\ &= \inf\{\} = \sup \mathbb{T} = 1. \end{aligned}$$

Hence,

$$\sigma(\omega) = \begin{cases} \frac{\omega}{1-\omega}, & \omega \neq 1, \\ 1, & \omega = 1. \end{cases}$$

Similarly, by using the definition of backward jump operator, $\rho(\omega) = \frac{\omega}{\omega+1}$.

Remark 1.2.1

- *For* $\omega = 0$, $\sigma(0) = 0 = \rho(0)$, $\Rightarrow 0 \in \mathbb{T}$ *is dense.*
- *For* $\omega = 1$, $\sigma(1) = 1$, $\Rightarrow 1 \in \mathbb{T}$ *is right dense.*
- *For* $\omega \neq 0, 1$; $\sigma(\omega) > \omega$, $\Rightarrow 0, 1 \neq \omega \in \mathbb{T}$ *is right scattered.*
- *For* $\omega \neq 0$, $\rho(\omega) < \omega$, $\Rightarrow 0 \neq \omega \in \mathbb{T}$ *is left scattered.*

Example 1.2.2 *Let* $\mathbb{T} = \{\frac{\omega}{2} : \omega \in \mathbf{N}_0\}$. *By definition* 1.2.2

$$\sigma\left(\frac{\mathfrak{s}}{2}\right) = \frac{\mathfrak{s}+1}{2} \Rightarrow \sigma(\omega) = \omega + \frac{1}{2}$$

and

$$\rho(\omega) = \begin{cases} \omega - \frac{1}{2}, & \omega \neq 0, \\ 0, & \omega = 0. \end{cases}$$

Definition 1.2.3 [*5*] *The graininess function* $\mu : \mathbb{T} \to [0, \infty)$ *is defined by*

$$\mu(\mathfrak{s}) := \sigma(\mathfrak{s}) - \mathfrak{s}.$$

Definition 1.2.4 [*5*] *If* $\mathbb{T}$ *has a left-scattered maximum* m, *then* $\mathbb{T}^k = \mathbb{T} - \{m\}$. *Otherwise,* $\mathbb{T}^k = \mathbb{T}$. *In summary*

$$\mathbb{T}^k = \begin{cases} \mathbb{T}\backslash(\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \sup \mathbb{T} < \infty \\ \mathbb{T}, & \sup \mathbb{T} = \infty. \end{cases}$$

Example 1.2.3 *Let* $\mathbb{T} := \{\frac{1}{\omega} : \omega \in \mathbf{N}\} \cup \{0\}$, *then* $\sup \mathbb{T} = 1$,

$$\begin{aligned} \mathbb{T}^k &= \mathbb{T}\backslash(\rho(\sup \mathbb{T}), \sup \mathbb{T}] \\ &= (\{1, \frac{1}{2}, \frac{1}{3}, \cdots\} \cup \{0\})\backslash(\frac{1}{2}, 1] \\ &= \{\frac{1}{2}, \frac{1}{3}, \cdots\} \cup \{0\}. \end{aligned}$$

In other words, $m = 1$ *and* $\mathbb{T}^k = (\{1, \frac{1}{2}, \frac{1}{3}, \cdots\} \cup \{0\}) - \{1\} = \{\frac{1}{2}, \frac{1}{3}, \cdots\} \cup \{0\}$.

Example 1.2.4 *Let* $\mathbb{T} := \{-\frac{1}{\omega} : \omega \in \mathbf{N}\} \cup \{\mathbf{N}_0\} = \{-1, -\frac{1}{2}, -\frac{1}{3}, \cdots\} \cup \{0, 1, 2, \cdots\}$, *then* $\sup \mathbb{T} = \infty$ *and* $\mathbb{T}^k = \mathbb{T}$.

Definition 1.2.5 *Let* $\mathfrak{f} : \mathbb{T} \to \mathrm{R}$ *be a function and* $\mathfrak{s} \in \mathbb{T}^k$*. Then for each* $\epsilon > 0$*, there exists a neighborhood* $\mathbf{U}$ *of* $\mathfrak{s}$ *such that*

$$|[\mathfrak{f}(\sigma(\mathfrak{s})) - \mathfrak{f}(\omega)] - \mathfrak{f}^{\Delta}(\mathfrak{s})[\sigma(\mathfrak{s}) - \omega]| \leq \epsilon|\sigma(\mathfrak{s}) - \omega|, \ \forall \ \omega \in \mathbf{U},$$

where $\mathfrak{f}^{\Delta}$ *denotes the derivative of* $\mathfrak{f}$ *with respect to* $\mathfrak{s}$*.*

Example 1.2.5 *If* $\mathfrak{f} : \mathbb{T} \to \mathrm{R}$ *is a function such that* $\mathfrak{f}(\mathfrak{s}) = \sqrt[3]{\mathfrak{s}}$*. By definition* 1.2.5

$$|[\sqrt[3]{\sigma(\mathfrak{s})} - \sqrt[3]{\omega}] - \mathfrak{f}^{\Delta}(\mathfrak{s})[\sigma(\mathfrak{s}) - \omega]| \leq \epsilon|\sigma(\mathfrak{s}) - \omega|, \ \forall \ \omega \in \mathbf{U}.$$

Which can also be written as

$$\begin{aligned}
&|[\sqrt[3]{\sigma(\mathfrak{s})} - \sqrt[3]{\omega}] - \mathfrak{f}^{\Delta}(\mathfrak{s})[(\sqrt[3]{\sigma(\mathfrak{s})} - \sqrt[3]{\omega}).((\sigma(\mathfrak{s}))^{\frac{2}{3}} + \sqrt[3]{\sigma(\mathfrak{s})}.\sqrt[3]{\omega} + \omega^{\frac{2}{3}})]| \\
&\leq \epsilon|(\sqrt[3]{\sigma(\mathfrak{s})} - \sqrt[3]{\omega}).((\sigma(\mathfrak{s}))^{\frac{2}{3}} + \sqrt[3]{\sigma(\mathfrak{s})}.\sqrt[3]{\omega} + \omega^{\frac{2}{3}})|, \ \forall \ \omega \in \mathbf{U} \\
&\Rightarrow |1 - \mathfrak{f}^{\Delta}(\mathfrak{s})[(\sigma(\mathfrak{s}))^{\frac{2}{3}} + \sqrt[3]{\sigma(\mathfrak{s})}.\sqrt[3]{\omega} + \omega^{\frac{2}{3}}]| \\
&\leq \epsilon|(\sigma(\mathfrak{s}))^{\frac{2}{3}} + \sqrt[3]{\sigma(\mathfrak{s})}.\sqrt[3]{\omega} + \omega^{\frac{2}{3}}|, \ \forall \ \omega \in \mathbf{U}.
\end{aligned}$$

Hence,

$$\mathfrak{f}^{\Delta}(\mathfrak{s}) = \frac{1}{(\sigma(\mathfrak{s}))^{\frac{2}{3}} + \sqrt[3]{\sigma(\mathfrak{s})}.\sqrt[3]{\mathfrak{s}} + \mathfrak{s}^{\frac{2}{3}}}.$$

- *For* $\mathbb{T} = \mathbf{R}$, $\mathfrak{f}^{\Delta}(\mathfrak{s}) = \frac{1}{3\mathfrak{s}^{\frac{2}{3}}} = \mathfrak{f}'(\mathfrak{s})$.
- *For* $\mathbb{T} = \mathbf{Z}$, $\mathfrak{f}^{\Delta}(\mathfrak{s}) = \frac{1}{(\mathfrak{s}+1)^{\frac{2}{3}} + \sqrt[3]{\mathfrak{s}+1}.\sqrt[3]{\mathfrak{s}} + \mathfrak{s}^{\frac{2}{3}}} = \Delta\mathfrak{f}(\mathfrak{s})$.
- *For* $\mathbb{T} := \{\sqrt[3]{\omega} : \omega \in \mathbf{N}_0\}$, $\mathfrak{f}^{\Delta}(\mathfrak{s}) = \frac{1}{(\mathfrak{s}^3+1)^{\frac{2}{9}} + \sqrt[9]{\mathfrak{s}^3+1}.\sqrt[3]{\mathfrak{s}} + \mathfrak{s}^{\frac{2}{3}}} = \Delta\mathfrak{f}(\mathfrak{s})$.

Definition 1.2.6 [5] *A function* $\mathfrak{f} : \mathbb{T} \to \mathbf{R}$ *is called rd-continuous provided it is continuous at right-dense points in* $\mathbb{T}$ *and its left-sided limit exist at left dense points in* $\mathbb{T}$*. The set of rd-continuous functions* $\mathfrak{f} : \mathbb{T} \to \mathbf{R}$ *is denoted by*

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbf{R}).$$

Remark 1.2.2 *Every continuous function is rd-continuous but every rd-continuous is not continuous.*

Example 1.2.6 *If* $\mathbb{T} = \mathbf{N}_0 \cup \{1 - \frac{1}{\omega} : \omega \in \mathbf{N}\}$ *and*

$$\mathfrak{f}(\mathfrak{s}) = \begin{cases} 0, & \mathfrak{s} \in \mathbf{N} \\ \mathfrak{s}, & \textit{Otherwise} \end{cases}$$

Then $\mathfrak{f}$ *is rd-continuous on* $\mathbb{T}$ *but not continuous.*

Definition 1.2.7 [5] *A function $q : \mathbb{T} \to \mathbf{R}$ is regressive provided*

$$1 + \mu(\mathfrak{s})q(\mathfrak{s}) \neq 0 \ \forall \mathfrak{s} \in \mathbb{T}^k$$

holds. The set of all regressive and rd-continuous functions $\mathfrak{f} : \mathbb{T} \to \mathbf{R}$ is denoted by

$$\mathfrak{R} = \mathfrak{R}(\mathbb{T}) = \mathfrak{R}(\mathbb{T}, \mathbf{R}).$$

Definition 1.2.8 *The Cylinder transformation $\xi_m(\mathbf{z})$ is defined as:*

$$\xi_m(\mathbf{z}) == \begin{cases} \frac{1}{\mathrm{m}} Log\left(1 + \mathrm{m}\mathbf{z}\right), & \textit{if } \mathrm{m} \neq 0 \quad (\textit{for} \ \ \mathbf{z} \neq -\frac{1}{\mathrm{m}}), \\ \mathbf{z}, & \textit{if } \mathrm{m} = 0. \end{cases}$$

Where Γ is the principle logarithm function.

Definition 1.2.9 [5] *If $q \in \mathfrak{R}$, then the exponential function $e_q : \mathbb{T} \times \mathbb{T} \to \mathbf{R}$ is defined as:*

$$e_q(\mathfrak{s}_2, \mathfrak{s}_1) = \exp\left(\int_{\mathfrak{s}_1}^{\mathfrak{s}_2} \xi_{\mu(\omega)}(q(\omega))\Delta\omega \right).$$

Remark 1.2.3

- *For $\mathbb{T} = \mathbf{R}$, $e_q(\mathfrak{s}_2, \mathfrak{s}_1) = \exp\left(\int_{\mathfrak{s}_1}^{\mathfrak{s}_2} q(\omega)d\omega \right)$.*
- *For $\mathbb{T} = \mathbf{Z}$, $e_q(\mathfrak{s}_2, \mathfrak{s}_1) = \prod_{\omega=\mathfrak{s}_1}^{\mathfrak{s}_2 - 1} \left(1 + q(\omega)\right)$.*

Theorem 1.2.7 [5] *(**Gronwall's Inequality**) Let $\mathfrak{r}_1, \mathfrak{r}_2 \in C_{rd}$ and $q \in \mathfrak{R}^+$, $q \geq 0$. Then*

$$\mathfrak{r}_1(\mathfrak{s}) \leq \mathfrak{r}_2(\mathfrak{s}) + \int_{\mathfrak{s}_0}^{\mathfrak{s}} \mathfrak{r}_1(\omega)q(\omega)\Delta\omega, \forall \omega \in \mathbb{T}$$

$$\Rightarrow \mathfrak{r}_1(\mathfrak{s}) \leq \mathfrak{r}_2(\mathfrak{s}) + \int_{\mathfrak{s}_0}^{\mathfrak{s}} e_q\left(\mathfrak{s}, \sigma(\omega)\right) \mathfrak{r}_2(w)q(\omega)\Delta\omega, \forall \omega \in \mathbb{T}$$

Theorem 1.2.8 [5] *If $q \in \mathfrak{R}$ and $\mathfrak{s}_i \in \mathbb{T}, \ 1 \leq i \leq 3$, then*

$$[e_q(\mathfrak{s}_3, .)]^{\Delta} = -q[e_q(\mathfrak{s}_3, .)]^{\sigma}$$

and

$$\int_{\mathfrak{s}_1}^{\mathfrak{s}_2} q(\omega)e_q(\mathfrak{s}_3, \sigma(\omega))\Delta\omega = e_q(\mathfrak{s}_3, \mathfrak{s}_1) - e_q(\mathfrak{s}_3, \mathfrak{s}_2).$$

Theorem 1.2.9 [5] *Let $\mathfrak{r}_1 : \mathbf{R} \to \mathbf{R}$ be continuously differentiable and suppose $\mathfrak{r}_2 : \mathbb{T} \to \mathbf{R}$ is delta differentiable. Then $\mathfrak{r}_1 \circ \mathfrak{r}_2 : \mathbb{T} \to \mathbf{R}$ is delta differentiable and*

$$(\mathfrak{r}_1 \circ \mathfrak{r}_2)^{\Delta}(\mathfrak{s}) = \left\{ \int_0^1 \mathfrak{r}_1^{'} \left(\mathfrak{r}_2(\mathfrak{s}) + h\mu(\mathfrak{s})\mathfrak{r}_2^{\Delta}(\mathfrak{s}) \right) dh \right\} \mathfrak{r}_2^{\Delta}(\mathfrak{s})$$

holds.

Example 1.2.10 *Let* $\mathbb{T} = \mathbf{N}_0 := \{\sqrt[3]{\omega} : \omega \in \mathbf{N}_0\}$, $\mathfrak{r}_1(\mathfrak{s}) = \sqrt[3]{\mathfrak{s}^2+6}$, $\mathfrak{r}_2(\mathfrak{s}) = \mathfrak{s}^3$. *Then* ${\mathfrak{r}_2}'(\mathfrak{s}) = 3\mathfrak{s}^2$. *Now,*

$$\begin{aligned}
(\mathfrak{r}_2 \circ \mathfrak{r}_1)(\mathfrak{s}) &= \mathfrak{r}_2(\mathfrak{r}_1(\mathfrak{s})) \\
&= \mathfrak{r}_2(\sqrt[3]{\mathfrak{s}^2+6}) \\
&= \mathfrak{s}^2 + 6 \\
\Rightarrow (\mathfrak{r}_2 \circ \mathfrak{r}_1)^{\Delta}(\mathfrak{s}) &= \mathfrak{s} + \sqrt[3]{\mathfrak{s}^3+1}
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{r}_1^{\Delta}(\mathfrak{s}) &= \frac{\mathfrak{r}_1(\sigma(\mathfrak{s})) - \mathfrak{r}_1(\mathfrak{s})}{\sigma(\mathfrak{s}) - \mathfrak{s}} \\
&= \frac{\mathfrak{r}_1(\sqrt[3]{\mathfrak{s}^3+1}) - \mathfrak{r}_1(\mathfrak{s})}{\sqrt[3]{\mathfrak{s}^3+1} - \mathfrak{s}} \\
&= \frac{\sqrt[3]{(\mathfrak{s}^3+1)^{\frac{2}{3}}+6} - \sqrt[3]{\mathfrak{s}^2+6}}{\sqrt[3]{\mathfrak{s}^3+1} - \mathfrak{s}}.
\end{aligned}$$

Also,

$$\begin{aligned}
&\int_0^1 \mathfrak{r}_2'\left(\mathfrak{r}_1(\mathfrak{s}) + h\mu(\mathfrak{s})\mathfrak{r}_1^{\Delta}(\mathfrak{s})\right) dh \\
&= \int_0^1 3[\sqrt[3]{\mathfrak{s}^2+6} + h\sqrt[3]{(\mathfrak{s}^3+1)^{\frac{2}{3}}+6} - \sqrt[3]{\mathfrak{s}^2+6}]^2 dh \\
&= \left| \frac{[\sqrt[3]{\mathfrak{s}^2+6} + h\sqrt[3]{(\mathfrak{s}^3+1)^{\frac{2}{3}}+6} - \sqrt[3]{\mathfrak{s}^2+6}]^3}{\sqrt[3]{(\mathfrak{s}^3+1)^{\frac{2}{3}}+6} - \sqrt[3]{\mathfrak{s}^2+6}} \right|_{h=0}^{h=1} \\
&= \frac{(\mathfrak{s}^3+1)^{\frac{2}{3}} - \mathfrak{s}^2}{\sqrt[3]{(\mathfrak{s}^3+1)^{\frac{2}{3}}+6} - \sqrt[3]{\mathfrak{s}^2+6}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\{\int_0^1 \mathfrak{r}_2'\left(\mathfrak{r}_1(\mathfrak{s}) + h\mu(\mathfrak{s})\mathfrak{r}_1^{\Delta}(\mathfrak{s})\right) dh\right\} \mathfrak{r}_1^{\Delta}(\mathfrak{s}) &= \frac{(\mathfrak{s}^3+1)^{\frac{2}{3}} - \mathfrak{s}^2}{\sqrt[3]{\mathfrak{s}^3+1} - \mathfrak{s}} \\
&= \mathfrak{s} + \sqrt[3]{\mathfrak{s}^3+1} \\
&= (\mathfrak{r}_2 \circ \mathfrak{r}_1)^{\Delta}(\mathfrak{s}).
\end{aligned}$$

Theorem 1.2.11 [5] *Let* $\mathfrak{r}_1 : \mathbb{T} \to \mathbf{R}$ *is strictly increasing and* $\overline{\mathbb{T}} := \mathfrak{r}_1(\mathbb{T})$ *is a time scale. Let* $\mathfrak{r}_2 : \overline{\mathbb{T}} \to \mathbf{R}$. *If* $\mathfrak{r}_1^{\Delta}(\mathfrak{s})$ *and* $\mathfrak{r}_2^{\overline{\Delta}}(\mathfrak{r}_1(\mathfrak{s}))$ *exist for* $\mathfrak{s} \in \mathbb{T}^k$, *then*

$$(\mathfrak{r}_2 \circ \mathfrak{r}_1)^{\Delta} = (\mathfrak{r}_2^{\overline{\Delta}} \circ \mathfrak{r}_1)\mathfrak{r}_1^{\Delta}.$$

Example 1.2.12 *Let* $\mathbb{T} = \mathbf{N}_0$, $\mathfrak{r}_1(\mathfrak{s}) = \mathfrak{s}^2$. $\mathfrak{r}_2(\mathfrak{s}) = 2\mathfrak{s}^2 + 5$. *Then* $\mathfrak{r}_1{}^{\Delta}(\mathfrak{s}) = 2\mathfrak{s} + 1$, $\overline{\mathbb{T}} := \mathfrak{r}_1(\mathbb{T}) = \{\omega^2 : \omega \in \mathbf{N}_0\}$. *Now,*

$$\begin{aligned}
(\mathfrak{r}_2 \circ \mathfrak{r}_1)(\mathfrak{s}) &= \mathfrak{r}_2(\mathfrak{r}_1(\mathfrak{s})) \\
&= \mathfrak{r}_2(\mathfrak{s}^2) \\
&= 2\mathfrak{s}^4 + 5 \\
\Rightarrow (\mathfrak{r}_2 \circ \mathfrak{r}_1)^{\Delta}(\mathfrak{s}) &= 2(2\mathfrak{s}+1)[\mathfrak{s}^2 + (\mathfrak{s}+1)^2]
\end{aligned}$$

and

$$\begin{aligned}
(\mathfrak{r}_2^{\overline{\Delta}} \circ \mathfrak{r}_1)(\mathfrak{s}) &= \mathfrak{r}_2{}^{\bar{\Delta}}(\mathfrak{s}^2) \\
&= \frac{\mathfrak{r}_2(\bar{\sigma}(\mathfrak{s}^2)) - \mathfrak{r}_2(\mathfrak{s}^2)}{\bar{\sigma}(\mathfrak{s}^2) - \mathfrak{s}^2} \\
&= \frac{\mathfrak{r}_2(\mathfrak{s}+1)^2 - \mathfrak{r}_2(\mathfrak{s}^2)}{(\mathfrak{s}+1)^2 - \mathfrak{s}^2} \\
&= 2[\mathfrak{s}^2 + (\mathfrak{s}+1)^2].
\end{aligned}$$

Thus, $(\mathfrak{r}_2^{\overline{\Delta}} \circ \mathfrak{r}_1)\mathfrak{r}_1^{\Delta} = 2[\mathfrak{s}^2 + (\mathfrak{s}+1)^2](2\mathfrak{s}+1) = (\mathfrak{r}_2 \circ \mathfrak{r}_1)^{\Delta}(\mathfrak{s})$.

1.3 Fractional Time Scales Essentials

Definition 1.3.1 [4] *Let* $\mathbb{T}$ *be a time scale such that* $\sup \mathbb{T} = \infty$ *and fix* $\mathfrak{s}_0 \in \mathbb{T}$. *For a given* $\mathfrak{f} : [\mathfrak{s}_0, \infty)_{\mathbb{T}} \to \mathbf{C}$, *the solution of the shifting problem*

$$\begin{aligned}
&\mathfrak{u}^{\Delta_{\mathfrak{s}}}(\mathfrak{s}, \sigma(\omega)) = -\mathfrak{u}^{\Delta_{\omega}}(\mathfrak{s}, \omega), && \mathfrak{s}, \omega \in \mathbb{T}, \mathfrak{s} \geq \omega \geq \mathfrak{s}_0 \\
&\mathfrak{u}(\mathfrak{s}, \mathfrak{s}_0) = \mathfrak{f}(\mathfrak{s}), && \mathfrak{s}, \omega \in \mathbb{T}, \mathfrak{s} \geq \mathfrak{s}_0
\end{aligned}$$

is denoted by $\widehat{\mathfrak{f}}$ *and is called the shift of* $\mathfrak{f}$.

Definition 1.3.2 [30] *The fractional generalized* $\Delta-$*power function* $h_\alpha : \mathbb{T} \times \mathbb{T} \to \mathbf{R}$ *on time scales is defined as:*

$$h_\alpha(\mathfrak{s}, \mathfrak{s}_0) = \mathcal{L}^{-1}\left\{\frac{1}{z^{\alpha+1}}\right\}(\mathfrak{s})$$

for those suitable regressive $z \in \mathbf{R}/\{0\}$ *such that* $\mathcal{L}^{-1}$ *exist for* $\alpha \in \mathbf{R}$, $\mathfrak{s} \geq \mathfrak{s}_0$. *Fractional generalized* $\Delta-$*power function* $h_\alpha(\mathfrak{s}, \omega)$ *on time scales is defined as the shift of* $h_\alpha(\mathfrak{s}, \mathfrak{s}_0)$, *that is,*

$$h_\alpha(\mathfrak{s}, \omega) = \widehat{h_\alpha(., \mathfrak{s}_0)}(\mathfrak{s}, \omega), \qquad \mathfrak{s} \geq \omega \geq \mathfrak{s}_0.$$

Definition 1.3.3 [*30*] *Let γ be a finite interval on a time scale $\mathbb{T}$ and $\mathfrak{s}_0, \mathfrak{s} \in \gamma$ such that $\mathfrak{s} > \mathfrak{s}_0$, then the Riemann-Liouville fractional $\Delta-$integral of $\mathfrak{f} : \mathbb{T} \to \mathbf{R}$, with order α is defined as:*

$$\left(I^{\alpha}_{\Delta,\mathfrak{s}_0}\mathfrak{f}\right)(\mathfrak{s}) = \begin{cases} \left(h_{\alpha-1}(.,\mathfrak{s}_0) * \mathfrak{f}\right)(\mathfrak{s}) = \int_{\mathfrak{s}_0}^{\mathfrak{s}} h_{\alpha-1}(\mathfrak{s},\sigma(\omega))\mathfrak{f}(\omega)\Delta\omega, & \alpha > 0; \\ \mathfrak{f}(\mathfrak{s}), & \alpha = 0. \end{cases}$$

Remark 1.3.1

- *For $\mathbb{T} = \mathbf{R}$, $\left(I^{\alpha}_{\Delta,\mathfrak{s}_0}\mathfrak{f}\right)(\mathfrak{s}) = \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{s}_0}^{\mathfrak{s}}(\mathfrak{s}-\omega)^{(\alpha-1)}\mathfrak{f}(\omega)d\omega = J^{\alpha}_{\mathfrak{s}_0+}$*
- *For $\mathbb{T} = \mathbf{Z}$, $\left(I^{\alpha}_{\Delta,\mathfrak{s}_0}\mathfrak{f}\right)(\mathfrak{s}) = \frac{1}{\Gamma(\alpha)}\sum_{\omega=\mathfrak{s}_0}^{\mathfrak{s}-1}(\mathfrak{s}-\omega-1)^{(\alpha-1)}\mathfrak{f}(\omega) = {}_{\mathfrak{s}_0}\Delta^{-\alpha}$*

Definition 1.3.4 [*30*] *Let $\alpha \geq 0$, $m = [\alpha]+1$ and $\mathfrak{f} : \mathbb{T} \to \mathbf{R}$. For $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathbb{T}^{k^m}$ with $\mathfrak{s}_1 < \mathfrak{s}_2$, the Riemann-Liouville fractional $\Delta-$derivative of order α is defined by*

$$D^{\alpha}_{\Delta,\mathfrak{s}_1}\mathfrak{f}(\mathfrak{s}_2) := D^{m}_{\Delta} I^{m-\alpha}_{\Delta,\mathfrak{s}_1}\mathfrak{f}(\mathfrak{s}_2),$$

if it exists.

Definition 1.3.5 [*30*] *Let $\alpha, \beta > 0$. The $\Delta-$Mittag-Leffler function ${}_{\Delta}F_{\alpha,\beta} : \mathbf{R} \times \mathbb{T} \times \mathbb{T} \to \mathbf{R}$ is defined as:*

$${}_{\Delta}F_{\alpha,\beta}\left(\lambda,\mathfrak{s},\mathfrak{s}_0\right) = \sum_{i=0}^{\infty}\lambda^{i}h_{i\alpha+\beta-1}(\mathfrak{s},\mathfrak{s}_0), \quad \mathfrak{s} \geq \mathfrak{s}_0$$

provided that right series is convergent.

1.4 Stochastic Differential Equation Essentials

To answer the question in which situation, we model a problem into stochastic differential equation, we consider the following example.

Example 1.4.1 *Consider the simple population growth model* [*19*] :

$$\frac{dR}{d\mathfrak{s}} = c(\mathfrak{s})R(\mathfrak{s}),\ R(0) = R_0(constant)$$

where $R(\mathfrak{s})$ is the size of the population at time $\mathfrak{s}$, and $c(\mathfrak{s})$ is the relative rate of growth at time $\mathfrak{s}$. It might happen that $c(\mathfrak{s})$ is not completely known due to influence of fluctuations of environment such as temperature, salinity, dissolved oxygen level, PH level, un-ionized ammonia level, etc. effects the growth of shrimp.

The equation, we obtain by allowing randomness in the coefficients of a differential equation is called a *stochastic differential equation* [19].

Definition 1.4.1 [*19*] *If* Ψ *is a given set, then a* $\sigma-$*algebra* $\mathcal{G}$ *on* Ψ *is a family of subsets of* Ψ *with the following properties:*

(a1) $\phi \in \mathcal{G}$

(a2) $G \in \mathcal{G} \Rightarrow G^c \in \mathcal{G},$ *where* G^c *is the compliment of* G *in* Ψ.

(a3) $C_1, C_2, \cdots \in \mathcal{G} \Rightarrow C := \cup_{i=1}^{\infty} C_i \in \mathcal{G}$

The pair $(\Psi, \mathcal{G})$ *is called a measurable space.*

Definition 1.4.2 [*19*] *A probability measure* ρ *on a a measurable space* $(\Psi, \mathcal{G})$ *is a function* $\rho : \mathcal{G} \to [0, 1]$ *such that*

(b1) $\rho(\phi) = 0, \ \rho(\Psi) = 1$

(b2) *If* $C_1, C_2, \cdots \in \mathcal{G}$ *and* $\{C_i\}_{i=1}^{\infty}$ *is disjoint, then*

$$\rho\left(\cup_{i=1}^{\infty} C_i\right) = \sum_{i=1}^{\infty} \rho(C_i).$$

The triple $(\Psi, \mathcal{G}, \rho)$ *is called a probability space. It is called a complete probability space if* $\mathcal{G}$ *contains all subsets* G *of* Ψ *with* $\rho-$*outer measure zero,*
i-e with

$$\rho * (F) := \inf\{\rho(G) : G \in \mathcal{G}, \ F \subset G\} = 0.$$

Definition 1.4.3 [*19*] *Let* $\mathfrak{U}$ *be any family of subsets of* Ψ, *then there is a smallest* $\sigma-$*algebra* $\mathfrak{H}_{\mathfrak{U}}$ *containing* $\mathfrak{U}$. *The* $\mathfrak{H}_{\mathfrak{U}}$ *is called the* $\sigma-$*algebra generated by* $\mathfrak{U}$ *i-e*

$$\mathfrak{H}_{\mathfrak{U}} = \cap\{\mathfrak{H} : \mathfrak{H} \ \sigma - algebra \ of \ \Psi, \mathfrak{U} \subset \mathfrak{H}\}.$$

Example 1.4.2 *If* $\mathfrak{U}$ *is the collection of all open subsets of a topological space* Ψ, *then* $\mathcal{O} = \mathfrak{H}_{\mathfrak{U}}$ *is called the Borel* $\sigma-$*algebra on* Ψ *and the elements* $B \in \mathcal{O}$ *are called Borel sets.*

Definition 1.4.4 [*19*] *A stochastic process is a parametrized collection of random variables* $\{\mathcal{K}_{\mathfrak{s}}\}_{\mathfrak{s} \in T}$ *defined on a probability space* $(\Psi, \mathcal{G}, \rho)$ *and assuming values in* $\mathbf{R}^n$.

Definition 1.4.5 *Brownian motion is the physical phenomenon which is discovered by the Robert Brown in 1827. It is the random motion suspended by a small particle.*

Definition 1.4.6 *The standard Brownian motion is a stochastic process* $(\mathfrak{B}_{\mathfrak{s}})_{\mathfrak{s}\in\mathbf{R}^+}$, *such that*

(c1) $\mathfrak{B}_0 = 0$ *almost surely,*

(c2) $\mathfrak{B}_{\mathfrak{s}}$ *is continuous for all values of* $\mathfrak{s}$ *and almost surely,*

(c3) *For any finite sequence of times* $\mathfrak{s}_0 < \mathfrak{s}_1 < \cdots < \mathfrak{s}_n$, *the increments*

$$\mathfrak{B}_{\mathfrak{s}_1} - \mathfrak{B}_{\mathfrak{s}_0}, \mathfrak{B}_{\mathfrak{s}_2} - \mathfrak{B}_{\mathfrak{s}_1}, \cdots, \mathfrak{B}_{\mathfrak{s}_n} - \mathfrak{B}_{\mathfrak{s}_{n-1}},$$

are mutually independent random variables,

(c4) *For any given times* $0 \leq \mathfrak{d} < \mathfrak{s}$, $\mathfrak{B}_{\mathfrak{s}} - \mathfrak{B}_{\mathfrak{d}}$ *has the Gaussian distribution* $\mathcal{N}(0, \mathfrak{s} - \mathfrak{d})$.

Definition 1.4.7 [19] *Let* $\mathfrak{B}_{\mathfrak{s}}(\mathfrak{d})$ *be* $n-$*dimensional Brownian motion. Then we define* $\mathcal{G}_{\mathfrak{s}} = \mathcal{G}_{\mathfrak{s}}^{(n)}$ *to be the* $\sigma-$*algebra generated by the random variables* $\{\mathfrak{B}_i(\mathfrak{s}_0)\}_{1\leq i\leq n, 0\leq \mathfrak{s}_0\leq \mathfrak{s}}$. *In other words,* $\mathcal{G}_{\mathfrak{s}}$ *is the smallest* $\sigma-$*algebra containing all sets of the form*

$$\{\mathfrak{d} : \mathfrak{B}_{\mathfrak{s}_1}(\mathfrak{d}) \in G_1, \cdots, \mathfrak{B}_{\mathfrak{s}_k}(\mathfrak{d}) \in G_k\},$$

where $\mathfrak{s}_1, \mathfrak{s}_2, \cdots, \mathfrak{s}_k \leq \mathfrak{s}$ *and* $G_1, G_2, \cdots, G_k$ *are Borel sets.*
A function $\mathfrak{h}(\mathfrak{a})$ *will be* $\mathcal{G}_{\mathfrak{s}}-$*measurable [19] if and only if* $\mathfrak{h}$ *can be written as the pointwise a.e. limit of sums of functions of the form* $\mathfrak{g}_1(\mathfrak{B}_{\mathfrak{s}_1})\mathfrak{g}_2(\mathfrak{B}_{\mathfrak{s}_2})\cdots\mathfrak{g}_k(\mathfrak{B}_{\mathfrak{s}_k})$, *where* $\mathfrak{g}_1, \mathfrak{g}_2, \cdots, \mathfrak{g}_k$ *are bounded continuous functions and* $\mathfrak{s}_1, \mathfrak{s}_2, \cdots, \mathfrak{s}_k \leq \mathfrak{s}$.

Example 1.4.3 $\mathfrak{h}_1(\mathfrak{d}) = \mathfrak{B}_{\mathfrak{s}/2}(\mathfrak{d})$ *is* $\mathcal{G}_{\mathfrak{s}}-$*measurable because all* $\mathfrak{s}$ *values are less than* $\mathfrak{s}$, *but* $\mathfrak{h}_2(\mathfrak{d}) = \mathfrak{B}_{2\mathfrak{s}}(\mathfrak{d})$ *is not because it contains information in the "future"* $(2\mathfrak{s} > \mathfrak{s})$.

Definition 1.4.8 [19] *Let* $\{\mathcal{S}_{\mathfrak{s}}\}_{\mathfrak{s}\geq 0}$ *be an increasing family of* $\sigma-$*algebras of subsets of* Ψ. *A process*

$$\mathfrak{g}(\mathfrak{s}, \mathfrak{d}) : [0, \infty) \times \Psi \to \mathbf{R}^n$$

is called $\mathcal{S}_{\mathfrak{s}}-$*adapted if for each* $\mathfrak{s} \geq 0$ *the function*

$$\mathfrak{d} \to \mathfrak{g}(\mathfrak{s}, \mathfrak{d})$$

is $\mathcal{S}_{\mathfrak{s}}-$*measurable.*

Example 1.4.4 *The process* $\mathfrak{h}_1(\mathfrak{s},\mathfrak{d}) = \mathfrak{B}_{\mathfrak{s}/2}(\mathfrak{d})$ *is* $\mathcal{G}_{\mathfrak{s}}-$*adapted, but* $\mathfrak{h}_2(\mathfrak{s},\mathfrak{d}) = \mathfrak{B}_{2\mathfrak{s}}(\mathfrak{d})$ *is not.*

Definition 1.4.9 [19] *Let* $\mathcal{U} = \mathcal{U}(\mathrm{T}_1, \mathrm{T}_2)$ *be the class of functions*

$$\mathfrak{f}(\mathfrak{s},\mathfrak{d}) : [0,\infty) \times \Psi \to \mathbf{R}$$

such that

(d1) $(\mathfrak{s},\mathfrak{d}) \to \mathfrak{f}(\mathfrak{s},\mathfrak{d})$ *is* $\mathcal{O} \times \mathcal{G}-$ *measurable, where* $\mathcal{O}$ *denotes the Borel* $\sigma-$*algebra on* $[0,\infty)$.

(d2) $\mathfrak{f}(\mathfrak{s},\mathfrak{d})$ *is* $\mathcal{G}_{\mathfrak{s}}-$*adapted.*

(d3) $E\left[\int_{\mathrm{T}_1}^{\mathrm{T}_2} \mathfrak{f}(\mathfrak{s},\mathfrak{d})^2 d\mathfrak{s}\right] < \infty.$

Definition 1.4.10 [19] *The characteristic function of a random variable* $K : \Psi \to \mathbf{R}^n$ *is the function* $\phi_K : \mathbf{R}^n \to C$ *defined by*

$$\phi_K(u_1, \cdots, u_n) = E[\exp(i(u_1K_1 + \cdots + u_nKn))] = \int_{\mathbf{R}^n} \exp(i < u, k >).P[K \in dk],$$

where $< u, k >= u_1k_1 + \cdots + u_nk_n$.

Remark 1.4.1 *1.* ϕ_K *is the Fourier transform of K.*

2. The characteristic function of K determines the distribution of K uniquely.

Definition 1.4.11 [19] *A function* $\varphi \in \mathcal{U}$ *is called elementary if it has the form*

$$\varphi(\mathfrak{s},\mathfrak{d}) = \sum_j e_j(\mathfrak{d}).\mathcal{K}_{[\mathfrak{s}_j,\mathfrak{s}_{j+1})}(\mathfrak{s}),$$

where $\mathcal{K}$ *denotes the characteristic function.*

Remark 1.4.2 *Since* $\varphi \in \mathcal{U}$ *each function* e_j *must be* $\mathcal{G}_{\mathfrak{s}_j}-$*measurable.*

Lemma 1.4.5 [19] ***(Itô Isometry)*** *If* $\varphi(\mathfrak{s},\mathfrak{d})$ *is bounded and elementary, then*

$$E\left[\left(\int_{\mathrm{T}_1}^{\mathrm{T}_2} \varphi(\mathfrak{s},\mathfrak{d}) d\mathfrak{B}_{\mathfrak{s}}(\mathfrak{d})\right)^2\right] = E\left[\int_{\mathrm{T}_1}^{\mathrm{T}_2} \varphi(\mathfrak{s},\mathfrak{d})^2 d\mathfrak{s}\right].$$

Lemma 1.4.6 [1] ***(Borel-Cantelli Lemma)*** *Let* $\{G_n\}$ *be a sequence of events on a probability space* $(\Psi, \mathcal{G}, \rho)$. *Then*

(e1) *if* $\sum_{n=1}^{\infty}\rho(G_n)<\infty,$ *then* $\rho(\limsup G_n)=0;$

(e2) *if* $\{G_n\}$ *are independent and*

$$\sum_{n=1}^{\infty}\rho(G_n)=\infty,$$

then $\rho(\limsup G_n)=1,$ *where* $\limsup G_n=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}G_k.$

Definition 1.4.12 [19] *A filtration on* $(\Psi,\mathcal{G})$ *is a family* $\mathcal{D}=\{\mathcal{D}_{\mathfrak{s}}\}_{\mathfrak{s}\geq 0}$ *of a* $\sigma-$*algebras* $\mathcal{D}_{\mathfrak{s}}\subset\mathcal{G}$ *such that* $0\leq\mathfrak{s}_0<\mathfrak{s}\Rightarrow\mathcal{D}_{\mathfrak{s}_0}\subset\mathcal{D}_{\mathfrak{s}}.$

Definition 1.4.13 [19] *An* $n-$*dimensional stochastic process* $\{\mathfrak{M}_{\mathfrak{s}}\}_{\mathfrak{s}\geq 0}$ *on* $(\Psi,\mathcal{G},\rho)$ *is called a martingale with respect to a filtration* $\{\mathcal{D}_{\mathfrak{s}}\}_{\mathfrak{s}\geq 0}$ *if*

(f1) $\mathfrak{M}_{\mathfrak{s}}$ *is* $\mathcal{D}_{\mathfrak{s}}-$*measurable for all* $\mathfrak{s}$,

(f2) $E[|\mathfrak{M}_{\mathfrak{s}}|]<\infty$ *for all* $\mathfrak{s}$,

(f3) $E[\mathfrak{M}_{\mathfrak{s}}|\mathcal{D}_{\mathfrak{s}_0}]=\mathfrak{M}_{\mathfrak{s}_0}$ *for all* $\mathfrak{s}\geq\mathfrak{s}_0.$

Example 1.4.7 [19] *Brownian motion* $\mathfrak{B}_{\mathfrak{s}}$ *in* $\mathbf{R}^n$ *is a martingale with respect to the* $\sigma-$*algebras* $\mathcal{G}_{\mathfrak{s}}$ *generated by* $\{\mathfrak{B}_{\mathfrak{s}_0}:\mathfrak{s}_0\leq\mathfrak{s}\}.$

Theorem 1.4.8 [19] *If* $\mathfrak{M}_{\mathfrak{s}}$ *is a martingale such that* $\mathfrak{s}\to\mathfrak{M}_{\mathfrak{s}}(\mathfrak{c})$ *is continuous a.s., then for* $\mathfrak{q}\geq 1,$ $T\geq 0$ *and* $w>0$

$$P\left(\sup_{0\leq\mathfrak{s}\leq T}|\mathfrak{M}_{\mathfrak{s}}|\geq w\right)\leq\frac{1}{w^{\mathfrak{q}}}.E[|\mathfrak{M}_T|^{\mathfrak{q}}].$$

1.5 Some Basic Essentials

Lemma 1.5.1 [17] *Let* $a_1,a_2>0,$ $s_1,s_2\in\mathbf{R}$ *and* $s_1\neq s_2$. *Then*

$$\int_{s_1}^{s_2}(s_2-\mathfrak{d})^{a_1-1}(\mathfrak{d}-s_1)^{a_2-1}d\mathfrak{d}=(s_2-s_1)^{a_1+a_2-1}\frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(a_1+a_2)}.$$

Corollary 1.5.2 [29]***(Cauchy Schwartz inequality)*** *Let* $\mathfrak{f}_1,\mathfrak{f}_2\in\lceil([s_1,s_2],\mathbf{R}),$ *then*

$$\left(\int_{s_1}^{s_2}\mathfrak{f}_1(\mathfrak{d})\mathfrak{f}_2(\mathfrak{d})d\mathfrak{d}\right)^2\leq\int_{s_1}^{s_2}\left(\mathfrak{f}_1(\mathfrak{d})\right)^2d\mathfrak{d}.\int_{s_1}^{s_2}\left(\mathfrak{f}_2(\mathfrak{d})\right)^2d\mathfrak{d}.$$

Lemma 1.5.3 [*17*] *Let* $a_1, a_2 \in R$. *Then for* $\xi > 0$, *we have*

$$\frac{\Gamma(\xi + a_1)}{\Gamma(\xi + a_2)} = O\left(\xi^{a_1 - a_2}\right), \quad \xi \to \infty.$$

Definition 1.5.1 [*17*] *Let* $a_2 > a_1 > 0, \quad \varrho > 0$. *Then the following definition:*

$$F_{\varrho, a_1, a_2}(\xi) := \sum_{n=0}^{\infty} b_n \xi^n, \quad \xi \in \mathbf{R}$$

is well defined, where b_0 *is a positive constant, and* $b_{n+1} = \left(\frac{\Gamma(n\varrho + a_1)}{\Gamma(n\varrho + a_2)}\right) b_n$.

Lemma 1.5.4 [*14*] *Let* $\mathfrak{s} \geq 0$; $\mathfrak{e}_1 \geq \mathfrak{e}_2 \geq 0$, *with* $\mathfrak{e}_1 \neq 0$. *Then, for any* $\mathfrak{a} > 0$

$$\mathfrak{s}^{\frac{\mathfrak{e}_2}{\mathfrak{e}_1}} \leq \frac{\mathfrak{e}_2}{\mathfrak{e}_1} \mathfrak{a}^{\frac{\mathfrak{e}_2 - \mathfrak{e}_1}{\mathfrak{d}_1}} \mathfrak{s} + \frac{\mathfrak{e}_1 - \mathfrak{e}_2}{\mathfrak{e}_1} \mathfrak{a}^{\frac{\mathfrak{e}_2}{\mathfrak{e}_1}}.$$

Lemma 1.5.5 [*18*] *Let* $\mathfrak{f}_1, \mathfrak{f}_2 \in C_{rd}(\mathbb{T}, \mathbf{R})$ *and* $q \in \mathfrak{R}^+$. *Then*

$$\mathfrak{f}_1^{\Delta}(\mathfrak{d}) \leq q(\mathfrak{d})\mathfrak{f}_1(\mathfrak{d}) + \mathfrak{f}_2(\mathfrak{d}),$$

$$\Rightarrow \mathfrak{f}_1(\mathfrak{d}) \leq \mathfrak{f}_1(\mathfrak{d}_0) \exp_q(\mathfrak{d}, \mathfrak{d}_0) + \int_{\mathfrak{d}_0}^{\mathfrak{d}} \exp_q(\mathfrak{d}, \sigma(\mathfrak{s}))\mathfrak{f}_2(\mathfrak{s})\Delta\mathfrak{s}.$$

Theorem 1.5.6 ***(Bernoulli's inequality)*** *For a real number* $\mathbf{w} > -1$ *and* $0 < \mathfrak{n} \leq 1$,

$$(1 + \mathbf{w})^{\mathfrak{n}} \leq 1 + \mathfrak{n}\mathbf{w}.$$

CHAPTER 2

Grownwall-Bellman type fractional and dynamical Integral Inequalities

It is well known that Grownwall-Bellman type inequalities play a significant role in the study of the boundedness, uniqueness and continuous dependence on the solutions of differential, integral and integro-differential equations. The following chapter is divided into three Sections. Section 2.1 is a motivation of an idea given by Q-X Kong et al. [17]. In Section 2.2, we investigate some delay integral inequalities on time scales to generalize, extend some existing results and to unify their corresponding discrete analogue. In Section 2.3, we construct generalized fractional Grownwall-Bellman type inequalities on time scales and give some definitions to structure the fractional $\Delta-$stochastic differential equation of It$\hat{o}-$Doob type on time scales.

2.1 Generalized Fractional Integral Inequality

Theorem 2.1.1 [*22*] *Let* $g_1(\mathfrak{x})$ *be a non-negative and locally integrable function on* $\mathbf{R}^+$; *let* $g_2(\mathfrak{x}), g_3(\mathfrak{x})$ *are nonnegative, nondecreasing continuous functions defined on* $\mathbf{R}^+$ *and bounded. Further, if* $r(\mathfrak{x})$ *is a nonnegative and* $\mathfrak{x}^{a-1}r(\mathfrak{x})$ *is locally integrable* *on* $\mathbf{R}^+$ *such that:*

$$r(\mathfrak{x}) \leq g_1(\mathfrak{x}) + g_2(\mathfrak{x}) \int_0^{\mathfrak{x}} (\mathfrak{x}-p)^{b-1} p^{a-1} r(p)dp + g_3(\mathfrak{x}) \int_0^{\mathfrak{x}} \mathfrak{x}^{b-1} p^{a-1} r(p)dp, \ \mathfrak{x} \in \mathbf{R}^+, \tag{2.1.1}$$

Then, for each constant $a > 0;\ 0 < b < 1;\ c = a+b-1 > 0;\ \omega > 0;\ \mathfrak{x} \in [0,\omega];\ \theta, \eta \in N$, *we have*

$$r(\mathfrak{x}) \leq \begin{cases} g_1(\mathfrak{x}) + \sum_{\theta=1}^{\infty} (\Gamma(b))^{\theta-1} \prod_{i=1}^{\theta-1} \frac{\Gamma(ic)}{\Gamma(ic+b)} \\ \times \sum_{\eta=0}^{\theta} C_\eta^\theta g_2^{\theta-\eta}(\mathfrak{x}) g_3^\eta(\mathfrak{x}) \int_0^{\mathfrak{x}} (\mathfrak{x}-p)^{\theta c-a} p^{a-1} g_1(p)dp, \quad a, b \in (0,1)\ ; \\ \\ g_1(\mathfrak{x}) + \sum_{\theta=1}^{\infty} \frac{(\Gamma(b))^\theta \ \mathfrak{x}^{(\theta-1)(a-1)}}{\Gamma(\theta b)} \\ \times \sum_{\eta=0}^{\theta} C_\eta^\theta g_2^{\theta-\eta}(\mathfrak{x}) g_3^\eta(\mathfrak{x}) \int_0^{\mathfrak{x}} (\mathfrak{x}-p)^{\theta b-1} p^{a-1} g_1(p)dp, a \in [1,\infty), b \in (0,1). \end{cases} \tag{2.1.2}$$

Proof. The proof of the inequality (2.1.1) would be followed by two cases. In the first case, we may assume $a, b \in (0,1)$ and in the second case, we may assume that $a \in [1,\infty)$ and $b \in (0,1)$.
On letting

$$\mathfrak{A} r(\mathfrak{x}) := g_2(\mathfrak{x}) \int_0^{\mathfrak{x}} (\mathfrak{x}-p)^{b-1} p^{a-1} r(p)dp + g_3(\mathfrak{x}) \int_0^{\mathfrak{x}} \mathfrak{x}^{b-1} p^{a-1} r(p)dp.$$

In this case, (2.1.1) is reshaped as:

$$r(\mathfrak{x}) \leq g_1(\mathfrak{x}) + \mathfrak{A} r(\mathfrak{x})$$

Iterating the inequality for some $\theta \in N$, one has

$$r(\mathfrak{x}) \leq \sum_{\eta=0}^{\theta-1} \mathfrak{A}^\eta g_1(\mathfrak{x}) + \mathfrak{A}^\theta r(\mathfrak{x}) \tag{2.1.3}$$

We claim that the following inequality does hold:

$$\mathfrak{A}^\theta r(\mathfrak{x}) \leq \begin{cases} (\Gamma(b))^{\theta-1} \prod_{i=1}^{\theta-1} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=0}^{\theta} C_\eta^\theta g_2^{\theta-\eta}(\mathfrak{x}) g_3^\eta(\mathfrak{x}) \\ \times \int_0^{\mathfrak{x}} (\mathfrak{x}-p)^{\theta c-a} p^{a-1} r(p)dp, \qquad a, b \in (0,1)\ ; \\ \\ \frac{(\Gamma(b))^\theta \mathfrak{x}^{(\theta-1)(a-1)}}{\Gamma(\theta b)} \sum_{\eta=0}^{\theta} C_\eta^\theta g_2^{\theta-\eta}(\mathfrak{x}) g_3^\eta(\mathfrak{x}) \\ \times \int_0^{\mathfrak{x}} (\mathfrak{x}-p)^{\theta b-1} p^{a-1} r(p)dp, \qquad a \in [1,\infty), b \in (0,1), \end{cases} \tag{2.1.4}$$

for some $\theta \in N$, where $\prod_{i=1}^{0} g(i) = 1$.

Case-I: The proof follows the induction criteria on θ. For $\theta = 1$, consider

$$\begin{aligned}\mathfrak{A}r(\mathfrak{x}) &= g_2(\mathfrak{x})\int_0^{\mathfrak{x}}(\mathfrak{x}-p)^{b-1}p^{a-1}r(p)dp + g_3(\mathfrak{x})\int_0^{\mathfrak{x}}\mathfrak{x}^{b-1}p^{a-1}r(p)dp\\ &\leq (g_2(\mathfrak{x})+g_3(\mathfrak{x}))\int_0^{\mathfrak{x}}(\mathfrak{x}-p)^{b-1}p^{a-1}r(p)dp,\end{aligned}$$

which is true by virtue of $\prod_{i=1}^{0} g(i) = 1$.

Suppose it holds for some $\theta = m$. Then, for $\theta = m+1$

$$\begin{aligned}\mathfrak{A}^{m+1}r(\mathfrak{x}) &= \mathfrak{A}(\mathfrak{A}^m r(\mathfrak{x}))\\ &= g_2(\mathfrak{x})\int_0^{\mathfrak{x}}(\mathfrak{x}-p)^{b-1}p^{a-1}\mathfrak{A}^m r(p)dp + g_3(\mathfrak{x})\int_0^{\mathfrak{x}}\mathfrak{x}^{b-1}p^{a-1}\mathfrak{A}^m r(p)dp\\ &\leq g_2(\mathfrak{x})\int_0^{\mathfrak{x}}(\mathfrak{x}-p)^{b-1}p^{a-1}\left(\Gamma(b)\right)^{m-1}\prod_{i=1}^{m-1}\frac{\Gamma(ic)}{\Gamma(ic+b)}\sum_{\eta=0}^{m}C_\eta^m g_2^{m-\eta}(p)g_3^{\eta}(p)\\ &\quad\times\int_0^p(p-\zeta)^{mc-a}\zeta^{a-1}r(\zeta)d\zeta dp + g_3(\mathfrak{x})\int_0^{\mathfrak{x}}\mathfrak{x}^{b-1}p^{a-1}\left(\Gamma(b)\right)^{m-1}\\ &\quad\times\prod_{i=1}^{m-1}\frac{\Gamma(ic)}{\Gamma(ic+b)}\sum_{\eta=0}^{m}C_\eta^m g_2^{m-\eta}(p)g_3^{\eta}(p)\int_0^p(p-\zeta)^{mc-a}\zeta^{a-1}r(\zeta)d\zeta dp\\ &\leq \left(\Gamma(b)\right)^{m-1}\prod_{i=1}^{m-1}\frac{\Gamma(ic)}{\Gamma(ic+b)}\sum_{\eta=0}^{m}C_\eta^m g_2^{m-\eta+1}(\mathfrak{x})g_3^{\eta}(\mathfrak{x})\\ &\quad\times\int_0^{\mathfrak{x}}(\mathfrak{x}-p)^{b-1}p^{a-1}\int_0^p(p-\zeta)^{mc-a}\zeta^{a-1}r(\zeta)d\zeta dp\\ &\quad+\left(\Gamma(b)\right)^{m-1}\prod_{i=1}^{m-1}\frac{\Gamma(ic)}{\Gamma(ic+b)}\sum_{\eta=0}^{m}C_\eta^m g_2^{m-\eta}(\mathfrak{x})g_3^{\eta+1}(\mathfrak{x})\\ &\quad\times\int_0^{\mathfrak{x}}\mathfrak{x}^{b-1}p^{a-1}\int_0^p(p-\zeta)^{mc-a}\zeta^{a-1}r(\zeta)d\zeta dp\end{aligned}$$

Change of order of integration yields the following:

$$\begin{aligned}\mathfrak{A}^{m+1}r(\mathfrak{x}) &\leq \left(\Gamma(b)\right)^{m-1}\prod_{i=1}^{m-1}\frac{\Gamma(ic)}{\Gamma(ic+b)}\sum_{\eta=0}^{m}C_\eta^m g_2^{m-\eta+1}(\mathfrak{x})g_3^{\eta}(\mathfrak{x})\int_0^{\mathfrak{x}}\zeta^{a-1}r(\zeta)\\ &\quad\times\int_\zeta^{\mathfrak{x}}(\mathfrak{x}-p)^{b-1}p^{a-1}(p-\zeta)^{mc-a}dpd\zeta + \left(\Gamma(b)\right)^{m-1}\prod_{i=1}^{m-1}\frac{\Gamma(ic)}{\Gamma(ic+b)}\\ &\quad\times\sum_{\eta=0}^{m}C_\eta^m g_2^{m-\eta}(\mathfrak{x})g_3^{\eta+1}(\mathfrak{x})\int_0^{\mathfrak{x}}\zeta^{a-1}r(\zeta)\\ &\quad\times\int_\zeta^{\mathfrak{x}}(\mathfrak{x}-p)^{b-1}p^{a-1}(p-\zeta)^{mc-a}dpd\zeta\\ &\leq \left(\Gamma(b)\right)^{m-1}\prod_{i=1}^{m-1}\frac{\Gamma(ic)}{\Gamma(ic+b)}\sum_{\eta=0}^{m}C_\eta^m g_2^{m-\eta+1}(\mathfrak{x})g_3^{\eta}(\mathfrak{x})\int_0^{\mathfrak{x}}\zeta^{a-1}r(\zeta)\end{aligned}$$

$$\begin{aligned}
&\times \int_{\zeta}^{\mathfrak{x}} (\mathfrak{x}-p)^{b-1}(p-\zeta)^{mc-1} dp d\zeta + (\Gamma(b))^{m-1} \prod_{i=1}^{m-1} \frac{\Gamma(ic)}{\Gamma(ic+b)} \\
&\times \sum_{\eta=0}^{m} C_{\eta}^{m} g_2^{m-\eta}(\mathfrak{x}) g_3^{\eta+1}(\mathfrak{x}) \int_0^{\mathfrak{x}} \zeta^{a-1} r(\zeta) \int_{\zeta}^{\mathfrak{x}} (\mathfrak{x}-p)^{b-1}(p-\zeta)^{mc-1} dp d\zeta \\
= \; & (\Gamma(b))^{m-1} \prod_{i=1}^{m-1} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=0}^{m} C_{\eta}^{m} g_2^{m-\eta+1}(\mathfrak{x}) g_3^{\eta}(\mathfrak{x}) \int_0^{\mathfrak{x}} \zeta^{a-1} r(\zeta) \\
&\times \frac{\Gamma(b)\Gamma(mc)}{\Gamma(b+mc)} (\mathfrak{x}-\zeta)^{b+mc-1} d\zeta + (\Gamma(b))^{m-1} \prod_{i=1}^{m-1} \frac{\Gamma(ic)}{\Gamma(ic+b)} \\
&\times \sum_{\eta=0}^{m} C_{\eta}^{m} g_2^{m-\eta}(\mathfrak{x}) g_3^{\eta+1}(\mathfrak{x}) \int_0^{\mathfrak{x}} \zeta^{a-1} r(\zeta) \frac{\Gamma(b)\Gamma(mc)}{\Gamma(b+mc)} (\mathfrak{x}-\zeta)^{b+mc-1} d\zeta \\
= \; & (\Gamma(b))^{m} \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=0}^{m} C_{\eta}^{m} g_2^{m-\eta+1}(\mathfrak{x}) g_3^{\eta}(\mathfrak{x}) \\
&\times \int_0^{\mathfrak{x}} (\mathfrak{x}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta + (\Gamma(b))^{m} \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} \\
&\times \sum_{\eta=0}^{m} C_{\eta}^{m} g_2^{m-\eta}(\mathfrak{x}) g_3^{\eta+1}(\mathfrak{x}) \int_0^{\mathfrak{x}} (\mathfrak{x}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta \\
= \; & (\Gamma(b))^{m} \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} C_0^m g_2^{m+1}(\mathfrak{x}) \int_0^{\mathfrak{x}} (\mathfrak{x}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta \\
&+ (\Gamma(b))^{m} \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=1}^{m} C_{\eta}^{m} g_2^{m-\eta+1}(\mathfrak{x}) g_3^{\eta}(\mathfrak{x}) \\
&\times \int_0^{\mathfrak{x}} (\mathfrak{x}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta + (\Gamma(b))^{m} \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} \\
&\times \sum_{\eta=1}^{m} C_{\eta-1}^{m} g_2^{m-\eta+1}(\mathfrak{x}) g_3^{\eta}(\mathfrak{x}) \int_0^{\mathfrak{x}} (\mathfrak{x}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta \\
&+ (\Gamma(b))^{m} \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} C_m^m g_3^{m+1}(\mathfrak{x}) \int_0^{\mathfrak{x}} (\mathfrak{x}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta \\
= \; & (\Gamma(b))^{m} \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} C_0^{m+1} g_2^{m+1}(\mathfrak{x}) \int_0^{\mathfrak{x}} (\mathfrak{x}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta \\
&+ (\Gamma(b))^{m} \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=1}^{m} \left(C_{\eta}^{m} + C_{\eta-1}^{m} \right) g_2^{m-\eta+1}(\mathfrak{x}) g_3^{\eta}(\mathfrak{x}) \\
&\times \int_0^{\mathfrak{x}} (\mathfrak{x}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta \\
&+ (\Gamma(b))^{m} \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} C_{m+1}^{m+1} g_3^{m+1}(\mathfrak{x}) \int_0^{\mathfrak{x}} (\mathfrak{x}-\zeta)^{(m+1)c-a} \zeta^{a-1} r(\zeta) d\zeta \\
= \; & (\Gamma(b))^{m} \prod_{i=1}^{m} \frac{\Gamma(ic)}{\Gamma(ic+b)} \sum_{\eta=0}^{m+1} C_{\eta}^{m+1} g_2^{m-\eta+1}(\mathfrak{x}) g_3^{\eta}(\mathfrak{x})
\end{aligned}$$

$$\times \int_0^{\mathfrak{x}} (\mathfrak{x}-\zeta)^{(m+1)c-a}\zeta^{a-1}r(\zeta)d\zeta,$$

which is no more than inequality (2.1.4) for $\theta = m+1$.

Case-II: For $\theta = 1$, the steps are same as $a, b \in (0,1)$.

Suppose (2.1.4) holds for some $\theta = m$. Then, for $\theta = m+1$, consider

$$\begin{aligned}
\mathfrak{A}^{m+1}r(\mathfrak{x}) &= \mathfrak{A}(\mathfrak{A}^m r(\mathfrak{x})) \\
&= g_2(\mathfrak{x})\int_0^{\mathfrak{x}} (\mathfrak{x}-p)^{b-1}p^{a-1}\mathfrak{A}^m r(p)dp + g_3(\mathfrak{x})\int_0^{\mathfrak{x}} \mathfrak{x}^{b-1}p^{a-1}\mathfrak{A}^m r(p)dp \\
&\leq g_2(\mathfrak{x})\int_0^{\mathfrak{x}} (\mathfrak{x}-p)^{b-1}p^{a-1}\frac{(\Gamma(b))^m\, p^{(m-1)(a-1)}}{\Gamma(mb)}\sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta}(p)g_3^{\eta}(p) \\
&\quad \times \int_0^{p} (p-\zeta)^{mb-1}\zeta^{a-1}r(\zeta)d\zeta dp \\
&\quad + g_3(\mathfrak{x})\int_0^{\mathfrak{x}} \mathfrak{x}^{b-1}p^{a-1}\frac{(\Gamma(b))^m\, p^{(m-1)(a-1)}}{\Gamma(mb)} \\
&\quad \times \sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta}(p)g_3^{\eta}(p)\int_0^{p} (p-\zeta)^{mb-1}\zeta^{a-1}r(\zeta)d\zeta dp \\
&\leq \frac{(\Gamma(b))^m}{\Gamma(mb)}\sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta+1}(\mathfrak{x})g_3^{\eta}(\mathfrak{x})\int_0^{\mathfrak{x}} (\mathfrak{x}-p)^{b-1}p^{a-1}p^{(m-1)(a-1)} \\
&\quad \times \int_0^{p} (p-\zeta)^{mb-1}\zeta^{a-1}r(\zeta)d\zeta dp + \frac{(\Gamma(b))^m}{\Gamma(mb)}\sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta}(\mathfrak{x})g_3^{\eta+1}(\mathfrak{x}) \\
&\quad \times \int_0^{\mathfrak{x}} \mathfrak{x}^{b-1}p^{a-1}p^{(m-1)(a-1)}\int_0^{p} (p-\zeta)^{mb-1}\zeta^{a-1}r(\zeta)d\zeta dp \\
&\leq \frac{(\Gamma(b))^m\, \mathfrak{x}^{m(a-1)}}{\Gamma(mb)}\sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta+1}(\mathfrak{x})g_3^{\eta}(\mathfrak{x})\int_0^{\mathfrak{x}} (\mathfrak{x}-p)^{b-1} \\
&\quad \times \int_0^{p} (p-\zeta)^{mb-1}\zeta^{a-1}r(\zeta)d\zeta dp + \frac{(\Gamma(b))^m\, \mathfrak{x}^{m(a-1)}}{\Gamma(mb)} \\
&\quad \times \sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta}(\mathfrak{x})g_3^{\eta+1}(\mathfrak{x})\int_0^{\mathfrak{x}} \mathfrak{x}^{b-1}\int_0^{p} (p-\zeta)^{mb-1}\zeta^{a-1}r(\zeta)d\zeta dp
\end{aligned}$$

Interchanging the order of integration yields

$$\begin{aligned}
\mathfrak{A}^{m+1}r(\mathfrak{x}) &\leq \frac{(\Gamma(b))^m\, \mathfrak{x}^{m(a-1)}}{\Gamma(mb)}\sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta+1}(\mathfrak{x})g_3^{\eta}(\mathfrak{x})\int_0^{\mathfrak{x}} \zeta^{a-1}r(\zeta) \\
&\quad \times \int_\zeta^{\mathfrak{x}} (\mathfrak{x}-p)^{b-1}(p-\zeta)^{mb-1}dpd\zeta + \frac{(\Gamma(b))^m\, \mathfrak{x}^{m(a-1)}}{\Gamma(mb)} \\
&\quad \times \sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta}(\mathfrak{x})g_3^{\eta+1}(\mathfrak{x})\int_0^{\mathfrak{x}} \zeta^{a-1}r(\zeta)\int_\zeta^{\mathfrak{x}} (\mathfrak{x}-p)^{b-1}(p-\zeta)^{mb-1}dpd\zeta \\
&= \frac{(\Gamma(b))^m\, \mathfrak{x}^{m(a-1)}}{\Gamma(mb)}\sum_{\eta=0}^{m} C_\eta^m g_2^{m-\eta+1}(\mathfrak{x})g_3^{\eta}(\mathfrak{x})\int_0^{\mathfrak{x}} \zeta^{a-1}r(\zeta)
\end{aligned}$$

$$
\begin{aligned}
&\times\frac{\Gamma(b)\Gamma(mb)}{\Gamma(b+mb)}(\mathfrak{x}-\zeta)^{b+mb-1}d\zeta+\frac{(\Gamma(b))^{m}\,\mathfrak{x}^{m(a-1)}}{\Gamma(mb)}\\
&\times\sum_{\eta=0}^{m}C_{\eta}^{m}g_{2}^{m-\eta}(\mathfrak{x})g_{3}^{\eta+1}(\mathfrak{x})\int_{0}^{\mathfrak{x}}\zeta^{a-1}r(\zeta)\frac{\Gamma(b)\Gamma(mb)}{\Gamma(b+mb)}(\mathfrak{x}-\zeta)^{b+mb-1}d\zeta\\
=\;&\frac{(\Gamma(b))^{m+1}\,\mathfrak{x}^{m(a-1)}}{\Gamma((m+1)b)}\sum_{\eta=0}^{m}C_{\eta}^{m}g_{2}^{m-\eta+1}(\mathfrak{x})g_{3}^{\eta}(\mathfrak{x})\\
&\times\int_{0}^{\mathfrak{x}}(\mathfrak{x}-\zeta)^{(m+1)b-1}\zeta^{a-1}r(\zeta)d\zeta+\frac{(\Gamma(b))^{m+1}\,\mathfrak{x}^{m(a-1)}}{\Gamma((m+1)b)}\\
&\times\sum_{\eta=0}^{m}C_{\eta}^{m}g_{2}^{m-\eta}(\mathfrak{x})g_{3}^{\eta+1}(\mathfrak{x})\int_{0}^{\mathfrak{x}}(\mathfrak{x}-\zeta)^{(m+1)b-1}\zeta^{a-1}r(\zeta)d\zeta\\
=\;&\frac{(\Gamma(b))^{m+1}\,\mathfrak{x}^{m(a-1)}}{\Gamma((m+1)b)}\sum_{\eta=0}^{m+1}C_{\eta}^{m+1}g_{2}^{m-\eta+1}(\mathfrak{x})g_{3}^{\eta}(\mathfrak{x})\\
&\times\int_{0}^{\mathfrak{x}}(\mathfrak{x}-\zeta)^{(m+1)b-1}\zeta^{a-1}r(\zeta)d\zeta
\end{aligned}
$$

which is no more than inequality (2.1.4) for $\theta = m+1$. We further, claim that $\mathfrak{A}^{\theta}r(\mathfrak{x}) \to 0$ as $\theta \to \infty$. Now, we go back to inequality (2.1.4).

For the case $a, b \in (0,1)$, there exists $N_1 > 0$ such that for $\theta > N_1$, we have

$$\theta c - a > 0,$$

and hence for an arbitrary $\omega > 0$

$$(\mathfrak{x}-p)^{\theta c-a} \le \omega^{\theta c-a}, \quad \mathfrak{x} \in [0,\omega], \; p \in [0,\mathfrak{x}].$$

Therefore, for $\theta > N_1$ and $\mathfrak{x} \in [0,\omega]$, we have

$$
\begin{aligned}
\mathfrak{A}^{\theta}r(\mathfrak{x}) &\le (\Gamma(b))^{\theta-1}\prod_{i=1}^{\theta-1}\frac{\Gamma(ic)}{\Gamma(ic+b)}\sum_{\eta=0}^{\theta}C_{\eta}^{\theta}g_{2}^{\theta-\eta}(\mathfrak{x})g_{3}^{\eta}(\mathfrak{x})\int_{0}^{\mathfrak{x}}(\mathfrak{x}-p)^{\theta c-a}p^{a-1}r(p)dp\\
&\le (\Gamma(b))^{\theta-1}\prod_{i=1}^{\theta-1}\frac{\Gamma(ic)}{\Gamma(ic+b)}\left(g_{2}(\mathfrak{x})+g_{3}(\mathfrak{x})\right)^{\theta}\int_{0}^{\mathfrak{x}}\omega^{\theta c-a}p^{a-1}r(p)dp \qquad (2.1.5)\\
&\le (\Gamma(b))^{\theta-1}\prod_{i=1}^{\theta-1}\frac{\Gamma(ic)}{\Gamma(ic+b)}\left(g_{2}(\mathfrak{x})+g_{3}(\mathfrak{x})\right)^{\theta}\omega^{\theta c-a}\int_{0}^{\omega}p^{a-1}r(p)dp
\end{aligned}
$$

For

$$\mathfrak{B}_{\theta} := (\Gamma(b))^{\theta-1}\prod_{i=1}^{\theta-1}\frac{\Gamma(ic)}{\Gamma(ic+b)}\left(g_{2}(\mathfrak{x})+g_{3}(\mathfrak{x})\right)^{\theta}\omega^{\theta c-a}.$$

Since $g_2(\mathfrak{x})$ and $g_3(\mathfrak{x})$ are bounded, so by Lemma 1.5.3

$$\frac{\mathfrak{B}_{\theta+1}}{\mathfrak{B}_{\theta}} = \frac{\Gamma(b)\Gamma(\theta c)}{\Gamma(\theta c+b)}\left(g_{2}(\mathfrak{x})+g_{3}(\mathfrak{x})\right)\omega^{c} \to 0 \;\; as\ \theta \to \infty$$

$p^{a-1}r(p)$ is locally integrable over R^+, so

$$\mathfrak{A}^\theta r(\mathfrak{x}) \to 0 \quad as \quad \theta \to \infty.$$

Similarly, we can prove that for $\theta > N_2$ and $\mathfrak{x} \in [0, \omega]$,

$$\begin{aligned}\sum_{\theta=1}^{\infty} \mathfrak{A}^\theta g_1(\mathfrak{x}) &= \sum_{\theta=1}^{N_2} \mathfrak{A}^\theta g_1(\mathfrak{x}) + \sum_{\theta=N_2+1}^{\infty} \mathfrak{A}^\theta g_1(\mathfrak{x}) \\ &\leq \sum_{\theta=1}^{N_2} \mathfrak{A}^\theta g_1(\mathfrak{x}) + \sum_{\theta=N_2+1}^{\infty} \mathfrak{B}_\theta \int_0^\omega p^{a-1} r(p)dp \\ &< \infty\end{aligned}$$

In a similar fashion, in Case-II, some one can prove $\mathfrak{A}^\theta r(\mathfrak{x}) \stackrel{\theta\to\infty}{\longrightarrow} 0$ and convergence of $\sum_{\theta=1}^{\infty} \mathfrak{A}^\theta g_1(\mathfrak{x})$ for $\mathfrak{x} \in [0, \omega]$. □

For $g_1(\mathfrak{x}) = g\mathfrak{x}^{d-1}$ in Theorem 2.1.1, the following holds.

Corollary 2.1.2 [22] *Let $a, d > 0$; $0 < b < 1$; $c = a + b - 1 > 0$; $e = a + d - 1 > 0$; $g > 0$; $g_2(\mathfrak{x})$ and $g_3(\mathfrak{x})$ are nonnegative, nondecreasing, bounded and continuous functions defined on $\mathbf{R}^+$. Further, suppose that $r(t)$ is a nonnegative and $\mathfrak{x}^{a-1}r(\mathfrak{x})$ is locally integrable on $\mathbf{R}^+$ such that:*

$$r(\mathfrak{x}) \leq g\mathfrak{x}^{d-1} + g_2(\mathfrak{x}) \int_0^{\mathfrak{x}} (\mathfrak{x} - p)^{b-1} p^{a-1} r(p)dp + g_3(\mathfrak{x}) \int_0^{\mathfrak{x}} \mathfrak{x}^{b-1} p^{a-1} r(p)dp, \ t \in \mathbf{R}^+, \tag{2.1.6}$$

Then

$$r(\mathfrak{x}) \leq g\mathfrak{x}^{d-1} F_{c,e,b+e} \left(\Gamma(b) \left(g_2(\mathfrak{x}) + g_3(\mathfrak{x})\right) \mathfrak{x}^c\right), \quad \mathfrak{x} \in \mathbf{R}^+. \tag{2.1.7}$$

Proof. From the proof of Theorem 2.1.1, we have $\mathfrak{A}^\theta r(\mathfrak{x}) \to 0$ as $\theta \to \infty$ for the cases $a, b \in (0, 1)$ and $a \in [1, \infty)$, $b \in (0, 1)$. This, together with (2.1.3), leads to

$$r(\mathfrak{x}) \leq \sum_{\eta=0}^{\infty} \left(\mathfrak{A}^\eta g\mathfrak{x}^{d-1}\right)(\mathfrak{x}).$$

Now, we show that

$$\left(\mathfrak{A}^\eta g\mathfrak{x}^{d-1}\right)(\mathfrak{x}) \leq g\mathfrak{x}^{d-1} \left(\mathfrak{x}^c\ \Gamma(b)\right)^\eta \prod_{i=0}^{\eta-1} \frac{\Gamma(ic + e)}{\Gamma(b + ic + e)} \sum_{i=0}^{\eta} C_i^\eta g_2^{\eta-i}(\mathfrak{x}) g_3^i(\mathfrak{x}), \ \eta \in N(2.1.8)$$

For $\theta = 0$, the result holds by virtue of $\prod_{i=0}^{\eta-1} g(i) = 1$. Suppose it holds for some $\theta = \eta$. For $\theta = \eta + 1$, one has

$$\left(\mathfrak{A}^\eta g\mathfrak{x}^{d-1}\right)(\mathfrak{x}) \;=\; g_2(\mathfrak{x}) \int_0^{\mathfrak{x}} (\mathfrak{x} - p)^{b-1} p^{a-1} \left(\mathfrak{A}^\eta g p^{d-1}\right)(p)dp$$

$$\begin{aligned}
& +g_3(\mathfrak{x}) \int_0^{\mathfrak{x}} \mathfrak{x}^{b-1} p^{a-1} \left(\mathfrak{A}^{\eta} g p^{d-1}\right)(p) dp \\
\leq\ & g_2(\mathfrak{x}) \int_0^{\mathfrak{x}} (\mathfrak{x}-p)^{b-1} p^{a-1} g p^{d-1} \left(p^c\ \Gamma(b)\right)^{\eta} \prod_{i=0}^{\eta-1} \frac{\Gamma(ic+e)}{\Gamma(b+ic+e)} \\
& \times \sum_{i=0}^{\eta} C_i^{\eta} g_2^{\eta-i}(p) g_3^{i}(p) dp + g_3(\mathfrak{x}) \int_0^{\mathfrak{x}} \mathfrak{x}^{b-1} p^{a-1} g p^{d-1} \left(p^c\ \Gamma(b)\right)^{\eta} \\
& \times \prod_{i=0}^{\eta-1} \frac{\Gamma(ic+e)}{\Gamma(b+ic+e)} \sum_{i=0}^{\eta} C_i^{\eta} g_2^{\eta-i}(p) g_3^{i}(p) dp \\
\leq\ & g\left(\Gamma(b)\right)^{\eta} \prod_{i=0}^{\eta-1} \frac{\Gamma(ic+e)}{\Gamma(b+ic+e)} \sum_{i=0}^{\eta} C_i^{\eta} g_2^{\eta-i+1}(\mathfrak{x}) g_3^{i}(\mathfrak{x}) \\
& \times \int_0^{\mathfrak{x}} (\mathfrak{x}-p)^{b-1} p^{a-1} p^{d-1} p^{\eta c} dp + g\left(\Gamma(b)\right)^{\eta} \prod_{i=0}^{\eta-1} \frac{\Gamma(ic+e)}{\Gamma(b+ic+e)} \\
& \times \sum_{i=0}^{\eta} C_i^{\eta} g_2^{\eta-i}(\mathfrak{x}) g_3^{i+1}(\mathfrak{x}) \int_0^{\mathfrak{x}} \mathfrak{x}^{b-1} p^{a-1} p^{d-1} p^{\eta c} dp \\
\leq\ & g\left(\Gamma(b)\right)^{\eta} \prod_{i=0}^{\eta-1} \frac{\Gamma(ic+e)}{\Gamma(b+ic+e)} \sum_{i=0}^{\eta+1} C_i^{\eta+1} g_2^{\eta-i+1}(\mathfrak{x}) g_3^{i}(\mathfrak{x}) \\
& \times \int_0^{\mathfrak{x}} (\mathfrak{x}-p)^{b-1} p^{a+d+\eta c-2} dp \\
=\ & g\left(\Gamma(b)\right)^{\eta} \prod_{i=0}^{\eta-1} \frac{\Gamma(ic+e)}{\Gamma(b+ic+e)} \sum_{i=0}^{\eta+1} C_i^{\eta+1} g_2^{\eta-i+1}(\mathfrak{x}) g_3^{i}(\mathfrak{x}) \\
& \times \frac{\Gamma(b)\Gamma(a+d+\eta c-1)}{\Gamma(a+b+d+\eta c-1)} \mathfrak{x}^{a+b+d+\eta c-2} \\
=\ & g\mathfrak{x}^{d-1} \left(\mathfrak{x}^{c}\Gamma(b)\right)^{\eta+1} \prod_{i=0}^{\eta} \frac{\Gamma(ic+e)}{\Gamma(b+ic+e)} \sum_{i=0}^{\eta+1} C_i^{\eta+1} g_2^{\eta-i+1}(\mathfrak{x}) g_3^{i}(\mathfrak{x})
\end{aligned}$$

Hence, inequality (2.1.8) is satisfied for any $\eta \in N$. In other words, we have proved that

$$r(\mathfrak{x}) \leq \sum_{\eta=0}^{\infty} g\mathfrak{x}^{d-1} \left(\mathfrak{x}^c\ \Gamma(b)\right)^{\eta} \prod_{i=0}^{\eta-1} \frac{\Gamma(ic+e)}{\Gamma(b+ic+e)} \sum_{i=0}^{\eta} C_i^{\eta} g_2^{\eta-i}(\mathfrak{x}) g_3^{i}(\mathfrak{x}).$$

By definition 1.5.1

$$r(\mathfrak{x}) \leq g\mathfrak{x}^{d-1} F_{c,e,b+e}\left(\Gamma(b)\left(g_2(\mathfrak{x})+g_3(\mathfrak{x})\right)\mathfrak{x}^{c}\right).$$

□

Remark 2.1.1 *For* $g_3(\mathfrak{x}) \equiv 0, \mathfrak{x} > 0$, *Corollary* 2.1.2 *reduces to* [*17, Theorem* 2.7] *for* $b \in (0,1)$.

2.2 Delay Double Integral Inequalities On Time Scales

In this section, we consider three types of integral inequalities on time scales which are given as follows:

$$\begin{aligned} w(u(\mathfrak{x}_1,\mathfrak{x}_2)) &\le a_1(\mathfrak{x}_1,\mathfrak{x}_2)+a_2(\mathfrak{x}_1,\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))\\ &\quad\times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_2(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))\\ &\quad+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ &\quad\times w_2(u(\mu_{1i}(\mathfrak{m}_1),\mu_{2i}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1 \end{aligned} \tag{2.2.1}$$

$$\begin{aligned} w(u(\mathfrak{x}_1,\mathfrak{x}_2)) &\le a_1(\mathfrak{x}_1,\mathfrak{x}_2)+a_2(\mathfrak{x}_1,\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))\\ &\quad\times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)w_2(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))\{w_3(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))\\ &\quad+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)w_3(u(\mu_{1i}(\mathfrak{m}_1),\mu_{2i}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\ &\quad+r_i(\mathfrak{t}_1,\mathfrak{t}_2)w_4(\log(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2))))]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1 \end{aligned} \tag{2.2.2}$$

$$\begin{aligned} w(u(\mathfrak{x}_1,\mathfrak{x}_2)) &\le a_1(\mathfrak{x}_1,\mathfrak{x}_2)+a_2(\mathfrak{x}_1,\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))\\ &\quad\times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_2(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))\\ &\quad+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)w_2(u(\mu_{1i}(\mathfrak{m}_1),\mu_{2i}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\ &\quad+r_i(\mathfrak{t}_1,\mathfrak{t}_2)\mathrm{L}(\mathfrak{t}_1,\mathfrak{t}_2,w_2(u(\mathfrak{t}_1,\mathfrak{t}_2)))]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1 \end{aligned} \tag{2.2.3}$$

with the initial condition, for $(\mathfrak{x}_1,\mathfrak{x}_2)\in\mathbb{T}_1\times\mathbb{T}_2$.

$$\begin{cases} w(u(\mathfrak{x}_1,\mathfrak{x}_2))=\mathfrak{a}(\mathfrak{x}_1,\mathfrak{x}_2), & \mathfrak{x}_1\in[\mathfrak{p}_1,\mathfrak{x}_{01}]_{\mathbb{T}} \text{ or } \mathfrak{x}_2\in[\mathfrak{p}_2,\mathfrak{x}_{02}]_{\mathbb{T}}\ ;\\ \mathfrak{a}(\mu_{1i}(\mathfrak{x}_1),\mu_{2i}(\mathfrak{x}_2))\le a_1(\mathfrak{x}_1,\mathfrak{x}_2), & \mu_{1i}(\mathfrak{x}_1)\le\mathfrak{x}_{01} \text{ or } \mu_{2i}(\mathfrak{x}_2)\le\mathfrak{x}_{02}, 1\le i\le n \end{cases} \tag{2.2.4}$$

Throughout the discussion of this section, $\mathrm{R}_0^+ := [0,\infty)$, $\mathrm{R}_1^+ := [1,\infty)$, $\mathfrak{x}_{0j}\in\mathbb{T}$, $\mathbb{T}_j=[\mathfrak{x}_{0j},\infty)_{\mathbb{T}}\subseteq\mathbb{T}^k$, $\mathbb{A}_j\subseteq\mathbf{N}_0$; $1\le j\le 2$, ρ_{ji} is the backward jump operator, $X^{\Delta y_i}(y_1,y_2,\cdots,y_n)$; $1\le i\le n$, is the partial delta-derivative of X with respect to $i-$th variable and $\Delta y_i X(y_1,y_2,\cdots,y_n)$ is the forward difference of X with respect to $i-$th variable. For $\mathfrak{r}>0$,

$$\mathfrak{G}_1(\mathfrak{r}) := \int_1^{\mathfrak{r}} \frac{\Delta p}{w_1(w^{-1}(p))} \quad \text{for} \quad \mathfrak{G}_1(\infty)=\infty.$$

$$\mathfrak{G}_{j+1}(\mathfrak{r}) := \int_1^{\mathfrak{r}} \frac{\Delta p}{w_{2^j}(w^{-1}(\mathfrak{G}_1^{-1}(p)))} \quad \text{for } \mathfrak{G}_{j+1}(\infty) = \infty.$$

$$\mathfrak{Q}_j(\mathfrak{r}) := \int_1^{\mathfrak{r}} \frac{\Delta p}{w_3(w^{-1}(\mathfrak{G}_1^{-1}(\mathfrak{G}_{j+1}^{-1}(p))))} \quad \text{for } \mathfrak{Q}_j(\infty) = \infty.$$

$$\mathfrak{b}_1(\mathfrak{x}_1,\mathfrak{x}_2) := \mathfrak{G}_1(a_1(\mathfrak{x}_1,\mathfrak{x}_2)) + a_2(\mathfrak{x}_1,\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} r_i(\mathfrak{t}_1,\mathfrak{t}_2)\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1.$$

$$\begin{aligned}\mathfrak{b}_2(\mathfrak{x}_1,\mathfrak{x}_2) &:= \mathfrak{G}_1(a_1(\mathfrak{x}_1,\mathfrak{x}_2)) + a_2(\mathfrak{x}_1,\mathfrak{x}_2)\\ &\times\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} r_i(\mathfrak{t}_1,\mathfrak{t}_2)\mathrm{L}(\mathfrak{t}_1,\mathfrak{t}_2,0)\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1.\end{aligned}$$

$$\begin{aligned}\mathfrak{c}(\mathfrak{x}_1,\mathfrak{x}_2) &:= \sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\ &\times(1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1)\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1.\end{aligned}$$

$$\mathfrak{d}_j(\mathfrak{x}_1,\mathfrak{x}_2) := \mathfrak{G}_{j+1}\left(\mathfrak{G}_1\left(a_1(\mathfrak{x}_1,\mathfrak{x}_2)\right)\right) + a_2(\mathfrak{x}_1,\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} r_i(\mathfrak{t}_1,\mathfrak{t}_2)\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1.$$

The following assumptions are to be considered for the convenient representation

(C1) $u, a_j, r_i \in C_{rd}(\mathbb{T}_1\times\mathbb{T}_2, \mathrm{R}_0^+)$; $\mu_{ji} \in (\mathbb{T}_j, \mathbb{T})$, $\gamma_{ji} \in C_{rd}^1(\mathbb{T}_j, \mathrm{R}_0^+)$; $w, w_k \in \mathcal{C}(\mathrm{R}_0^+, \mathrm{R}_0^+)$; $\mathfrak{a} \in C_{rd}\left(([\mathfrak{p}_1, \mathfrak{x}_{01}] \times [\mathfrak{p}_2, \mathfrak{x}_{02}])_{\mathbb{T}^2}, \mathrm{R}_0^+\right)$; $f_i, g_i, f_i^{\Delta\mathfrak{x}_1} \in C_{rd}(\mathbb{T}_1^2\times\mathbb{T}_2^2, \mathrm{R}_0^+)$, $1 \le k \le 4$.

(C2) a_j is nondecreasing in each variable.

(C3) w_k is nondecreasing function such that $w_k(p) > 0$ for $p > 0$.

(C4) w is nondecreasing with $\lim_{\mathrm{t}\to\infty} w(\mathrm{t}) = \infty$.

(C5) γ_{ji} is nondecreasing with $\gamma_{ji}(\mathfrak{x}_j) \le \mathfrak{x}_j$.

(C6) $\mu_{ji}(\mathfrak{x}_j) \le \mathfrak{x}_j$, $-\infty < \mathfrak{p}_j = \inf\{\min\left(\mu_{ji}(\mathfrak{x}_j)\right), \mathfrak{x}_j \in \mathbb{T}_j\} \le \mathfrak{x}_{0j}$.

Theorem 2.2.1 [*23*] *Let the inequalities* (2.2.1) *and* (2.2.4) *be hold. Then, under the conditions* (**C1**) – (**C6**) *for* $\mathfrak{x}_j \in \mathbb{T}_j$, $1 \le j,k \le 2$, *one has*

$$u(\mathfrak{x}_1,\mathfrak{x}_2) \le w^{-1}(\mathfrak{G}_1^{-1}(\mathfrak{G}_2^{-1}(\mathfrak{G}_2(\mathfrak{b}_1(\mathfrak{x}_1,\mathfrak{x}_2)) + a_2(\mathfrak{x}_1,\mathfrak{x}_2)\mathfrak{c}(\mathfrak{x}_1,\mathfrak{x}_2)))). \tag{2.2.5}$$

Proof. Under the condition (**C2**), the inequality (2.2.1) is rewritten as:

$$\begin{aligned} w(u(\mathfrak{x}_1,\mathfrak{x}_2)) &\le a_1(\mathfrak{y},\mathfrak{x}_2)+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))\\ &\quad\times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_2(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))\\ &\quad+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ &\quad\times w_2(u(\mu_{1i}(\mathfrak{m}_1),\mu_{2i}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1 \end{aligned}$$

where $(\mathfrak{x}_1,\mathfrak{x}_2)\in[\mathfrak{x}_{01},\mathfrak{y}]_{\mathbb{T}}\times\mathbb{T}_2$ for some fixed $\mathfrak{y}\in\mathbb{T}_1$.

On letting $\xi_1(\mathfrak{x}_1,\mathfrak{x}_2)$ by

$$\begin{aligned} \xi_1(\mathfrak{x}_1,\mathfrak{x}_2) &:= a_1(\mathfrak{y},\mathfrak{x}_2)+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))\\ &\quad\times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_2(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))\\ &\quad+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ &\quad\times w_2(u(\mu_{1i}(\mathfrak{m}_1),\mu_{2i}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1, \end{aligned} \tag{2.2.6}$$

then we have

$$\xi_1(\mathfrak{x}_{01},\mathfrak{x}_2)=a_1(\mathfrak{y},\mathfrak{x}_2), \tag{2.2.7}$$

and

$$u(\mathfrak{x}_1,\mathfrak{x}_2)\le w^{-1}(\xi_1(\mathfrak{x}_1,\mathfrak{x}_2)). \tag{2.2.8}$$

If $\mu_{1i}(\mathfrak{x}_1)\ge\mathfrak{x}_{01}$ and $\mu_{2i}(\mathfrak{x}_2)\ge\mathfrak{x}_{02}$, then

$$u(\mu_{1i}(\mathfrak{x}_1),\mu_{2i}(\mathfrak{x}_2))\le w^{-1}(\xi_1(\mu_{1i}(\mathfrak{x}_1),\mu_{2i}(\mathfrak{x}_2)))\le w^{-1}(\xi_1(\mathfrak{x}_1,\mathfrak{x}_2)). \tag{2.2.9}$$

On the other hand, if $\mu_{1i}(\mathfrak{x}_1)\le\mathfrak{x}_{01}$ or $\mu_{2i}(\mathfrak{x}_2)\le\mathfrak{x}_{02}$, then

$$\begin{aligned} u(\mu_{1i}(\mathfrak{x}_1),\mu_{2i}(\mathfrak{x}_2)) &= w^{-1}(\mathfrak{a}(\mu_{1i}(\mathfrak{x}_1),\mu_{2i}(\mathfrak{x}_2)))\\ &\le w^{-1}(a_1(\mathfrak{x}_1,\mathfrak{x}_2))\\ &\le w^{-1}(\xi_1(\mathfrak{x}_1,\mathfrak{x}_2)). \end{aligned} \tag{2.2.10}$$

From (2.2.9) and (2.2.10), we have

$$u(\mu_{1i}(\mathfrak{x}_1),\mu_{2i}(\mathfrak{x}_2))\le w^{-1}(\xi_1(\mathfrak{x}_1,\mathfrak{x}_2)) \text{ for } (\mathfrak{x}_1,\mathfrak{x}_2)\in[\mathfrak{x}_{01},\mathfrak{y}]_{\mathbb{T}}\times\mathbb{T}_2. \tag{2.2.11}$$

From (2.2.6), by [9, Lemma 1.2] we have

$$\xi_1^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) = a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(u(\mu_{1i}(\gamma_{1i}(\mathfrak{x}_1)),\mu_{2i}(\mathfrak{t}_2)))$$

$$\times[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\{w_2(u(\mu_{1i}(\gamma_{1i}(\mathfrak{x}_1)),\mu_{2i}(\mathfrak{t}_2)))$$
$$+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)$$
$$\times w_2(u(\mu_{1i}(\mathfrak{m}_1),\mu_{2i}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)]\Delta\mathfrak{t}_2$$
$$+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))$$
$$\times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_2(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))$$
$$+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)$$
$$\times w_2(u(\mu_{1i}(\mathfrak{m}_1),\mu_{2i}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}$$
$$+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1. \tag{2.2.12}$$

Plugging inequality (2.2.11) in (2.2.12), we have

$$\xi_1{}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \le a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(w^{-1}(\xi_1(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))$$
$$\times[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\{w_2(w^{-1}(\xi_1(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))$$
$$+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)$$
$$\times w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)]\Delta\mathfrak{t}_2$$
$$+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(w^{-1}(\xi_1(\mathfrak{t}_1,\mathfrak{t}_2)))$$
$$\times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_2(w^{-1}(\xi_1(\mathfrak{t}_1,\mathfrak{t}_2)))$$
$$+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)$$
$$\times w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}$$
$$+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1. \tag{2.2.13}$$

Using the fact that w_1, w, ξ_1 are nondecreasing, (2.2.13) become

$$\xi_1{}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \le w_1(w^{-1}(\xi_1(\mathfrak{x}_1,\mathfrak{x}_2)))a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)$$
$$\times\{w_2(w^{-1}(\xi_1(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)$$
$$\times w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)]\Delta\mathfrak{t}_2$$
$$+w_1(w^{-1}(\xi_1(\mathfrak{x}_1,\mathfrak{x}_2)))a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)$$
$$\times\{w_2(w^{-1}(\xi_1(\mathfrak{t}_1,\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)$$
$$\times w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1.$$

Equivalently,

$$\begin{aligned}
\frac{{\xi_1}^{\Delta \mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{w_1(w^{-1}(\xi_1(\mathfrak{x}_1,\mathfrak{x}_2)))} \leq\ & a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2) \\
& \times\{w_2(w^{-1}(\xi_1(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2))) \\
& +\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2) \\
& \times w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\} + r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)]\Delta\mathfrak{t}_2 \\
& +a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2) \\
& \times\{w_2(w^{-1}(\xi_1(\mathfrak{t}_1,\mathfrak{t}_2))) + \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2) \\
& \times w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\} + r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1 \\
=\ & a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}[\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2) \\
& \times\{w_2(w^{-1}(\xi_1(\mathfrak{t}_1,\mathfrak{t}_2))) + \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2) \\
& \times w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\} + r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1]^{\Delta\mathfrak{x}_1}.
\end{aligned}$$

Integrating over $[\mathfrak{x}_{01},\mathfrak{x}_1]$ and using the definition of $\mathfrak{G}_1$ yields:

$$\begin{aligned}
& \mathfrak{G}_1(\xi_1(\mathfrak{x}_1,\mathfrak{x}_2)) \\
\leq\ & \mathfrak{G}_1(\xi_1(\mathfrak{x}_{01},\mathfrak{x}_2)) + a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} r_i(\mathfrak{t}_1,\mathfrak{t}_2)\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1 \\
& +a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_2(w^{-1}(\xi_1(\mathfrak{t}_1,\mathfrak{t}_2))) \\
& +\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1 \\
\leq\ & \mathfrak{G}_1(a_1(\mathfrak{y},\mathfrak{x}_2)) + a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{y})}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} r_i(\mathfrak{t}_1,\mathfrak{t}_2)\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1 \\
& +a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_2(w^{-1}(\xi_1(t_1,\mathfrak{t}_2))) \\
& +\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1 \\
=\ & \mathfrak{b}_1(\mathfrak{y},\mathfrak{x}_2) + a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_2(w^{-1}(\xi_1(\mathfrak{t}_1,\mathfrak{t}_2))) \\
+ & \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1
\end{aligned}$$

On letting $\zeta_1(\mathfrak{x}_1,\mathfrak{x}_2)$ by

$$\begin{aligned}\zeta_1(\mathfrak{x}_1,\mathfrak{x}_2) &:= \mathfrak{b}_1(\mathfrak{y},\mathfrak{x}_2)+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\ &\times\{w_2(w^{-1}(\xi_1(\mathfrak{t}_1,\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ &\times w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1, \end{aligned} \tag{2.2.14}$$

then we have

$$\zeta_1(\mathfrak{x}_{01},\mathfrak{x}_2)=\mathfrak{b}_1(\mathfrak{y},\mathfrak{x}_2), \tag{2.2.15}$$

and

$$\xi_1(\mathfrak{x}_1,\mathfrak{x}_2)\le \mathfrak{G}_1^{-1}(\zeta_1(\mathfrak{x}_1,\mathfrak{x}_2)). \tag{2.2.16}$$

$$\begin{aligned}\Rightarrow {\zeta_1}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) &= a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\ &\times\{w_2(w^{-1}(\xi_1(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ &\times w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\\ &+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\ &\times\{w_2(w^{-1}(\xi_1(\mathfrak{t}_1,\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ &\times w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1. \end{aligned} \tag{2.2.17}$$

Plugging inequality (2.2.16) in (2.2.17), we have

$$\begin{aligned}{\zeta_1}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) &\le a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\ &\times\{w_2(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_1(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2))))\\ &+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ &\times w_2(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_1(\mathfrak{m}_1,\mathfrak{m}_2))))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\\ &+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\ &\times\{w_2(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_1(\mathfrak{t}_1,\mathfrak{t}_2))))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ &\times w_2(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_1(\mathfrak{m}_1,\mathfrak{m}_2))))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1. \end{aligned} \tag{2.2.18}$$

Using the fact that w_2, w, $\mathfrak{G}_1$, ζ_1 are nondecreasing, (2.2.18) become

$${\zeta_1}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \le w_2(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_1(\mathfrak{x}_1,\mathfrak{x}_2))))a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)$$

$$\times\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)$$
$$\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2$$
$$+w_2(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_1(\mathfrak{x}_1,\mathfrak{x}_2))))a_2(\mathfrak{y},\mathfrak{x}_2)$$
$$\times\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)$$
$$\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1.$$

Equivalently,

$$\frac{\zeta_1{}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{w_2(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_1(\mathfrak{x}_1,\mathfrak{x}_2))))}$$
$$\le a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)$$
$$\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2$$
$$+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)$$
$$\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1$$
$$= a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}[\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)$$
$$\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1]^{\Delta\mathfrak{x}_1}.$$

Integrating over $[\mathfrak{x}_{01},\mathfrak{x}_1]$ and using the definition of $\mathfrak{G}_2$ yields:

$$\begin{aligned}\mathfrak{G}_2(\zeta_1(\mathfrak{x}_1,\mathfrak{x}_2)) &\le \mathfrak{G}_2(\zeta_1(\mathfrak{x}_{01},\mathfrak{x}_2))+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\ &\quad\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\\ &= \mathfrak{G}_2(\mathfrak{b}_1(\mathfrak{y},\mathfrak{x}_2))+a_2(\mathfrak{y},\mathfrak{x}_2)\mathfrak{c}(\mathfrak{x}_1,\mathfrak{x}_2) \qquad (2.2.19)\end{aligned}$$

Combination of (2.2.8), (2.2.16) and (2.2.19) yields the desired result (2.2.5). □

Remark 2.2.1 • *For* $\mathbb{T}=\mathbf{Z}$, $a_1(\mathfrak{x}_1,\mathfrak{x}_2)\equiv c$, $a_2(\mathfrak{x}_1,\mathfrak{x}_2)\equiv 1\equiv w_2$, $\gamma_{ji}\equiv I\equiv\mu_{ji}$, $n=1$, $f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)=f(\mathfrak{t}_1,\mathfrak{t}_2)$, $g_i\equiv 0\equiv r_i$, *Theorem 2.2.1 coincides with [6, Theorem 2.1]. Moreover, for* $w(u)=u^p$, *it coincides with [7, Theorem 2.1].*

- *For* $\mathbb{T} = \mathbf{R}$, $a_1(\mathfrak{x}_1, \mathfrak{x}_2) \equiv c$, $a_2(\mathfrak{x}_1, \mathfrak{x}_2) \equiv 1$, $w_1 \equiv I \equiv \mu_{ji}$ $f_i(\mathfrak{x}_1, \mathfrak{t}_1, \mathfrak{x}_2, \mathfrak{t}_2) = f_i(\mathfrak{t}_1, \mathfrak{t}_2)$, $g_i \equiv 0$, $1 \le i \le n$, *Theorem 2.2.1 coincides with [8, Theorem 2.3]. Moreover, for* $r_i \equiv 0$, *it coincides with [8, Theorem 2.2].*

- *For* $\mathbb{T} = \mathbf{R}$, $w_1(u) = u^q$, $\mu_{ji} \equiv I$, $f_i(\mathfrak{x}_1, \mathfrak{t}_1, \mathfrak{x}_2, \mathfrak{t}_2) = f_i(\mathfrak{t}_1, \mathfrak{t}_2)$, $g_i \equiv 0$, $1 \le i \le n$, *Theorem 2.2.1 coincides with [16, Theorem 2.1].*

- *For* $n = 1$, $\gamma_{ji} \equiv I$, $w_2 \equiv 1$, $f_i(\mathfrak{x}_1, \mathfrak{t}_1, \mathfrak{x}_2, \mathfrak{t}_2) = f(\mathfrak{t}_1, \mathfrak{t}_2)$, $g_i \equiv 0 \equiv r_i$, *Theorem 2.2.1 coincides with [11, Theorem 1].*

- *For* $\mathbb{T} = \mathbf{Z}$, $w_2 \equiv 1$, $\gamma_{ji} \equiv I \equiv \mu_{ji}$, $n = 1$, $f_i(\mathfrak{x}_1, \mathfrak{t}_1, \mathfrak{x}_2, \mathfrak{t}_2) = f(\mathfrak{t}_1, \mathfrak{t}_2)$, $g_i \equiv 0 \equiv r_i$, *Theorem 2.2.1 coincides with [12, Theorem 1].*

- *For* $\mathbb{T} = \mathbf{R}$, $\mu_{ji} \equiv I$, $f_i(\mathfrak{x}_1, \mathfrak{t}_1, \mathfrak{x}_2, \mathfrak{t}_2) = f_i(\mathfrak{t}_1, \mathfrak{t}_2)$, $g_i \equiv 0$, *Theorem 2.2.1 coincides with [28, Theorem 1].*

Theorem 2.2.2 [23] *Let the inequalities* (2.2.2) *and* (2.2.4) *be hold and under the conditions* (**C1**)-(**C6**) *for* $u \in C_{rd}(\mathbb{T}_1 \times \mathbb{T}_2, \mathrm{R}_1^+)$, $\mathfrak{x}_j \in \mathbb{T}_j$, $1 \le j \le 2$

- *if* $w_2(\mathfrak{u}) \ge w_4(\log(\mathfrak{u}))$, *then*

$$u(\mathfrak{x}_1, \mathfrak{x}_2) \le w^{-1}(\mathfrak{G}_1^{-1}(\mathfrak{G}_2^{-1}(\mathfrak{Q}_1^{-1}(\mathfrak{Q}_1(\mathfrak{d}_1(\mathfrak{x}_1, \mathfrak{x}_2)) + a_2(\mathfrak{x}_1, \mathfrak{x}_2)\mathfrak{c}(\mathfrak{x}_1, \mathfrak{x}_2))))) \quad (2.2.20)$$

- *if* $w_2(\mathfrak{u}) < w_4(\log(\mathfrak{u}))$, *then*

$$u(\mathfrak{x}_1, \mathfrak{x}_2) \le w^{-1}(\mathfrak{G}_1^{-1}(\mathfrak{G}_3^{-1}(\mathfrak{Q}_2^{-1}(\mathfrak{Q}_2(\mathfrak{d}_2(\mathfrak{x}_1, \mathfrak{x}_2)) + a_2(\mathfrak{x}_1, \mathfrak{x}_2)\mathfrak{c}(\mathfrak{x}_1, \mathfrak{x}_2))))). \quad (2.2.21)$$

Proof. Under the condition (**C2**), the inequality (2.2.2) is rewritten as:

$$\begin{aligned} w(u(\mathfrak{x}_1, \mathfrak{x}_2)) \le \; & a_1(\mathfrak{y}, \mathfrak{x}_2) + a_2(\mathfrak{y}, \mathfrak{x}_2) \sum_{i=1}^{n} \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)} \int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(u(\mu_{1i}(\mathfrak{t}_1), \mu_{2i}(\mathfrak{t}_2))) \\ & \times [f_i(\mathfrak{x}_1, \mathfrak{t}_1, \mathfrak{x}_2, \mathfrak{t}_2) w_2(u(\mu_{1i}(\mathfrak{t}_1), \mu_{2i}(\mathfrak{t}_2))) \{ w_3(u(\mu_{1i}(\mathfrak{t}_1), \mu_{2i}(\mathfrak{t}_2))) \\ & + \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1} \int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1, \mathfrak{m}_1, \mathfrak{t}_2, \mathfrak{m}_2) w_3(u(\mu_{1i}(\mathfrak{m}_1), \mu_{2i}(\mathfrak{m}_2))) \Delta\mathfrak{m}_2 \Delta\mathfrak{m}_1 \} \\ & + r_i(\mathfrak{t}_1, \mathfrak{t}_2) w_4(\log(u(\mu_{1i}(\mathfrak{t}_1), \mu_{2i}(\mathfrak{t}_2))))] \Delta\mathfrak{t}_2 \Delta\mathfrak{t}_1, \end{aligned}$$

where $(\mathfrak{x}_1, \mathfrak{x}_2) \in [\mathfrak{x}_{01}, \mathfrak{y}]_{\mathbb{T}} \times \mathbb{T}_2$ for some fixed $\mathfrak{y} \in \mathbb{T}_1$.
On letting $\xi_2(\mathfrak{x}_1, \mathfrak{x}_2)$ by

$$\xi_2(\mathfrak{x}_1, \mathfrak{x}_2) \; := \; a_1(\mathfrak{y}, \mathfrak{x}_2) + a_2(\mathfrak{y}, \mathfrak{x}_2) \sum_{i=1}^{n} \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)} \int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(u(\mu_{1i}(\mathfrak{t}_1), \mu_{2i}(\mathfrak{t}_2)))$$

$$\times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)w_2(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))\{w_3(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))$$
$$+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)w_3(u(\mu_{1i}(\mathfrak{m}_1),\mu_{2i}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}$$
$$+r_i(\mathfrak{t}_1,\mathfrak{t}_2)w_4(\log(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2))))]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1, \tag{2.2.22}$$

then we have

$$\xi_2(\mathfrak{x}_{01},\mathfrak{x}_2)=a_1(\mathfrak{y},\mathfrak{x}_2), \tag{2.2.23}$$

and

$$u(\mathfrak{x}_1,\mathfrak{x}_2)\le w^{-1}(\xi_2(\mathfrak{x}_1,\mathfrak{x}_2)). \tag{2.2.24}$$

On going the identical steps from (2.2.8) – (2.2.10), one has

$$u(\mu_{1i}(\mathfrak{x}_1),\mu_{2i}(\mathfrak{x}_2))\le w^{-1}(\xi_2(\mathfrak{x}_1,\mathfrak{x}_2)) \text{ for } (\mathfrak{x}_1,\mathfrak{x}_2)\in[\mathfrak{x}_{01},\mathfrak{y}]_{\mathbb{T}}\times\mathbb{T}_2. \tag{2.2.25}$$

From (2.2.22), by [9, Lemma 1.2] we have

$$\begin{aligned}
\xi_2{}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \;=\;& a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(u(\mu_{1i}(\gamma_{1i}(\mathfrak{x}_1)),\mu_{2i}(\mathfrak{t}_2)))\\
&\times[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)w_2(u(\mu_{1i}(\gamma_{1i}(\mathfrak{x}_1)),\mu_{2i}(\mathfrak{t}_2)))\\
&\times\{w_3(u(\mu_{1i}(\gamma_{1i}(\mathfrak{x}_1)),\mu_{2i}(\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_3(u(\mu_{1i}(\mathfrak{m}_1),\mu_{2i}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)\\
&\times w_4(\log(u(\mu_{1i}(\gamma_{1i}(\mathfrak{x}_1)),\mu_{2i}(\mathfrak{t}_2))))]\Delta\mathfrak{t}_2\\
&+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))\\
&\times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)w_2(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))\{w_3(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))\\
&+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)w_3(u(\mu_{1i}(\mathfrak{m}_1),\mu_{2i}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&+r_i(\mathfrak{t}_1,\mathfrak{t}_2)w_4(\log(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2))))]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1.
\end{aligned} \tag{2.2.26}$$

Plugging inequality (2.2.25) in (2.2.26), we have

$$\begin{aligned}
\xi_2{}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \;=\;& a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(w^{-1}(\xi_2(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))\\
&\times[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)w_2(w^{-1}(\xi_2(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))\\
&\times\{w_3(w^{-1}(\xi_2(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_3(w^{-1}(\xi_2(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)\\
&\times w_4(\log(w^{-1}(\xi_2(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2))))]\Delta\mathfrak{t}_2\\
&+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))
\end{aligned}$$

$$\times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)w_2(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))\{w_3(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))$$
$$+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)w_3(w^{-1}(\xi_2(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}$$
$$+r_i(\mathfrak{t}_1,\mathfrak{t}_2)w_4(\log(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2))))]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1. \tag{2.2.27}$$

Using the fact that w_1, w, ξ_2 are nondecreasing, (2.2.27) become

$$\begin{aligned}
{\xi_2}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \leq\ & w_1(w^{-1}(\xi_2(\mathfrak{x}_1,\mathfrak{x}_2)))a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times w_2(w^{-1}(\xi_2(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))\{w_3(w^{-1}(\xi_2(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))\\
&+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)w_3(w^{-1}(\xi_2(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&+r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)w_4(\log(w^{-1}(\xi_2(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2))))]\Delta\mathfrak{t}_2\\
&+w_1(w^{-1}(\xi_2(\mathfrak{x}_1,\mathfrak{x}_2)))a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times w_2(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))\{w_3(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))\\
&+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)w_3(w^{-1}(\xi_2(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&+r_i(\mathfrak{t}_1,\mathfrak{t}_2)w_4(\log(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2))))]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
&\frac{{\xi_2}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{w_1(w^{-1}(\xi_2(\mathfrak{x}_1,\mathfrak{x}_2)))}\\
&\leq a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times w_2(w^{-1}(\xi_2(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))\{w_3(w^{-1}(\xi_2(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))\\
&+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)w_3(w^{-1}(\xi_2(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&+r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)w_4(\log(w^{-1}(\xi_2(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2))))]\Delta\mathfrak{t}_2\\
&+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times w_2(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))\{w_3(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))\\
&+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)w_3(w^{-1}(\xi_2(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&+r_i(\mathfrak{t}_1,\mathfrak{t}_2)w_4(\log(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2))))]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1\\
&= a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}[\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times w_2(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))\{w_3(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))\\
&+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)w_3(w^{-1}(\xi_2(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}
\end{aligned}$$

$$+r_i(\mathfrak{t}_1,\mathfrak{t}_2)w_4(\log(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2))))]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1]^{\Delta\mathfrak{x}_1}.$$

Integrating over $[\mathfrak{x}_{01},\mathfrak{x}_1]$ and using the definition of $\mathfrak{G}_1$ yields:

$$\begin{aligned}\mathfrak{G}_1(\xi_2(\mathfrak{x}_1,\mathfrak{x}_2)) \quad \leq \quad & \mathfrak{G}_1(\xi_2(\mathfrak{x}_{01},\mathfrak{x}_2))+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\ & \times w_2(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))\{w_3(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))\\ & +\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)w_3(w^{-1}(\xi_2(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\ & +r_i(\mathfrak{t}_1,\mathfrak{t}_2)w_4(\log(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2))))]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1. \end{aligned} \tag{2.2.28}$$

For $w_2(\mathfrak{u}) \geq w_4(\log(\mathfrak{u}))$, (2.2.28) is rewritten as:

$$\begin{aligned}\mathfrak{G}_1(\xi_2(\mathfrak{x}_1,\mathfrak{x}_2)) \quad \leq \quad & \mathfrak{G}_1(a_1(\mathfrak{y},\mathfrak{x}_2))+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_2(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))\\ & \times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_3(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))\\ & +\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ & \times w_3(w^{-1}(\xi_2(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1 \end{aligned} \tag{2.2.29}$$

On letting $\zeta_2(\mathfrak{x}_1,\mathfrak{x}_2)$ by

$$\begin{aligned}\zeta_2(\mathfrak{x}_1,\mathfrak{x}_2) \quad := \quad & \mathfrak{G}_1(a_1(\mathfrak{y},\mathfrak{x}_2))+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_2(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))\\ & \times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_3(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))\\ & +\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ & \times w_3(w^{-1}(\xi_2(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1, \end{aligned} \tag{2.2.30}$$

then we have

$$\zeta_2(\mathfrak{x}_{01},\mathfrak{x}_2)=\mathfrak{G}_1(a_1(\mathfrak{y},\mathfrak{x}_2)), \tag{2.2.31}$$

and

$$\xi_2(\mathfrak{x}_1,\mathfrak{x}_2)\leq \mathfrak{G}_1^{-1}\left(\zeta_2(\mathfrak{x}_1,\mathfrak{x}_2)\right). \tag{2.2.32}$$

$$\begin{aligned}\Rightarrow \zeta_2{}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \quad = \quad & a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_2(w^{-1}(\xi_2(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))\\ & \times[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\{w_3(w^{-1}(\xi_2(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))\\ & +\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ & \times w_3(w^{-1}(\xi_2(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)]\Delta\mathfrak{t}_2 \end{aligned}$$

$$+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_2(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)$$
$$\times\{w_3(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)$$
$$\times w_3(w^{-1}(\xi_2(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1 \quad (2.2.33)$$

Plugging inequality (2.2.32) in (2.2.33), we have

$$\begin{aligned}
{\zeta_2}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \leq\ & a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_2(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2))))\\
&\times[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\{w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2))))\\
&+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{m}_1,\mathfrak{m}_2))))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)]\Delta\mathfrak{t}_2\\
&+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_2(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{t}_1,\mathfrak{t}_2))))\\
&\times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{t}_1,\mathfrak{t}_2))))\\
&+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{m}_1,\mathfrak{m}_2))))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1. \qquad (2.2.34)
\end{aligned}$$

Using the fact that w_2, w, $\mathfrak{G}_1$, ζ_2 are nondecreasing, (2.2.34) become

$$\begin{aligned}
{\zeta_2}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \leq\ & w_2(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{x}_1,\mathfrak{x}_2))))a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\\
&\times\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\{w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2))))\\
&+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{m}_1,\mathfrak{m}_2))))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)]\Delta\mathfrak{t}_2\\
&+w_2(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{x}_1,\mathfrak{x}_2))))a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{t}_1,\mathfrak{t}_2))))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{m}_1,\mathfrak{m}_2))))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1.
\end{aligned}$$

Equivalently,

$$\frac{{\zeta_2}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{w_2(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{x}_1,\mathfrak{x}_2))))}$$

$$
\begin{aligned}
\leq\; & a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2))))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2}g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{m}_1,\mathfrak{m}_2))))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)]\Delta\mathfrak{t}_2\\
&+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{t}_1,\mathfrak{t}_2))))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2}g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{m}_1,\mathfrak{m}_2))))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1\\
=\; & a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}[\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{t}_1,\mathfrak{t}_2))))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2}g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{m}_1,\mathfrak{m}_2))))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1]^{\Delta\mathfrak{x}_1}.
\end{aligned}
$$

Integrating over $[\mathfrak{x}_{01},\mathfrak{x}_1]$ and using the definition of $\mathfrak{G}_2$ yields:

$$
\begin{aligned}
& \mathfrak{G}_2(\zeta_2(\mathfrak{x}_1,\mathfrak{x}_2))\\
\leq\; & \mathfrak{G}_2(\zeta_2(\mathfrak{x}_{01},\mathfrak{x}_2))+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{t}_1,\mathfrak{t}_2))))\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2}g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\\
\leq\; & \mathfrak{G}_2(\mathfrak{G}_1(a_1(\mathfrak{y},\mathfrak{x}_2)))+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{y})}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}r_i(\mathfrak{t}_1,\mathfrak{t}_2)\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\\
&+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{y})}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{t}_1,\mathfrak{t}_2))))\\
&\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2}g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\\
=\; & \mathfrak{d}_1(\mathfrak{y},\mathfrak{x}_2)+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_1)w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{t}_1,\mathfrak{t}_2))))\\
&\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2}g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1 \qquad (2.2.35)
\end{aligned}
$$

On letting $\upsilon(\mathfrak{x}_1,\mathfrak{x}_2)$ by

$$
\begin{aligned}
\upsilon(\mathfrak{x}_1,\mathfrak{x}_2) \;:=\; & \mathfrak{d}_1(\mathfrak{y},\mathfrak{x}_2)+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_1)\\
&\times w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{t}_1,\mathfrak{t}_2))))
\end{aligned}
$$

$$\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1, \tag{2.2.36}$$

then we have

$$\upsilon(\mathfrak{x}_{01},\mathfrak{x}_2)=\mathfrak{d}_1(\mathfrak{y},\mathfrak{x}_2), \tag{2.2.37}$$

and

$$\zeta_2(\mathfrak{x}_1,\mathfrak{x}_2)\leq\mathfrak{G}_2^{-1}(\upsilon(\mathfrak{x}_1,\mathfrak{x}_2)). \tag{2.2.38}$$

$$\begin{aligned}\Rightarrow \upsilon^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) &= a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\ &\times w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2))))\\ &\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\\ &+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_1)\\ &\times w_3(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_2(\mathfrak{t}_1,\mathfrak{t}_2))))\\ &\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1.\end{aligned} \tag{2.2.39}$$

Plugging inequality (2.2.38) in (2.2.39), we have

$$\begin{aligned}\upsilon^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) &= a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\ &\times w_3(w^{-1}(\mathfrak{G}_1^{-1}(\mathfrak{G}_2^{-1}(\upsilon(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))))\\ &\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\\ &+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_1)\\ &\times w_3(w^{-1}(\mathfrak{G}_1^{-1}(\mathfrak{G}_2^{-1}(\upsilon(\mathfrak{t}_1,\mathfrak{t}_2)))))\\ &\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1.\end{aligned} \tag{2.2.40}$$

Using the fact that w_3, w, $\mathfrak{G}_1$, $\mathfrak{G}_2$, υ are nondecreasing, (2.2.40) become

$$\begin{aligned}\upsilon^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) &\leq w_3(w^{-1}(\mathfrak{G}_1^{-1}(\mathfrak{G}_2^{-1}(\upsilon(\mathfrak{x}_1,\mathfrak{x}_2)))))\\ &\times a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}[\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\ &\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\end{aligned}$$

$$+w_3(w^{-1}(\mathfrak{G}_1^{-1}(\mathfrak{G}_2^{-1}(\upsilon(\mathfrak{x}_1,\mathfrak{x}_2)))))$$
$$\times a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_1)$$
$$\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1.$$

Equivalently,

$$\frac{\upsilon^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{w_3(w^{-1}(\mathfrak{G}_1^{-1}(\mathfrak{G}_2^{-1}(\upsilon(\mathfrak{x}_1,\mathfrak{x}_2)))))}$$
$$\le a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)$$
$$\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2$$
$$+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_1)$$
$$\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1$$
$$= a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}[\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_1)$$
$$\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1]^{\Delta\mathfrak{x}_1}.$$

Integrating over $[\mathfrak{x}_{01},\mathfrak{x}_1]$ and using the definition of $\mathfrak{Q}_1$, for $(\mathfrak{x}_1,\mathfrak{x}_2)\in[\mathfrak{x}_{01},\mathfrak{y}]_{\mathbb{T}}\times\mathbb{T}_2$, we have

$$\mathfrak{Q}_1(\upsilon(\mathfrak{x}_1,\mathfrak{x}_2)) \le \mathfrak{Q}_1(\mathfrak{d}_1(\mathfrak{y},\mathfrak{x}_2))+a_2(\mathfrak{y},\mathfrak{x}_2)\mathfrak{c}(\mathfrak{x}_1,\mathfrak{x}_2) \tag{2.2.41}$$

Combination of (2.2.24), (2.2.32), (2.2.38) and (2.2.41) yield the desired result (2.2.20).

For $w_2(\mathfrak{u}) < w_4(\log(\mathfrak{u}))$, (2.2.28) is rewritten as:

$$\mathfrak{G}_1(\xi_2(\mathfrak{x}_1,\mathfrak{x}_2))$$
$$\le \mathfrak{G}_1(\xi_2(\mathfrak{x}_{01},\mathfrak{x}_2))+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_4(\log(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2))))$$
$$\times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_3(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)$$
$$\times w_3(w^{-1}(\xi_2(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1$$
$$\le \mathfrak{G}_1(a_1(\mathfrak{y},\mathfrak{x}_2))+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_4(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))$$

$$\times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_3(w^{-1}(\xi_2(\mathfrak{t}_1,\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)$$
$$\times w_3(w^{-1}(\xi_2(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1 \qquad (2.2.42)$$

On going the identical steps from (2.2.30) – (2.2.41), we have

$$u(\mathfrak{x}_1,\mathfrak{x}_2)\le w^{-1}(\mathfrak{G}_1^{-1}(\mathfrak{G}_3^{-1}(\mathfrak{Q}_2^{-1}(\mathfrak{Q}_2(\mathfrak{d}_2(\mathfrak{x}_1,\mathfrak{x}_2))+a_2(\mathfrak{x}_1,\mathfrak{x}_2)\mathfrak{c}(\mathfrak{x}_1,\mathfrak{x}_2))))).$$

□

Remark 2.2.2 *For* $\mathbb{T}=\mathbf{R}$, $w_3\equiv 1$, $\mu_{ji}\equiv I$, $f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)=f_i(\mathfrak{t}_1,\mathfrak{t}_2)$, $g_i\equiv 0$, *Theorem 2.2.2 coincide with [28, Theorem 2].*

Theorem 2.2.3 [23] *Let the inequalities* (2.2.3) *and* (2.2.4) *be hold and under the conditions* (**C1**) – (**C6**). *If* $\mathrm{L},\mathrm{M}:\mathbb{T}_1\times\mathbb{T}_2\times\mathbf{R}_0^+\to\mathbf{R}_0^+$ *is right-dense continuous on* $\mathbb{T}_1\times\mathbb{T}_2$ *and continuous on* $\mathbf{R}_0^+$ *such that:*
$0\le \mathrm{L}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{u})-\mathrm{L}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{v})\le \mathrm{M}(\mathfrak{x}_1,\mathfrak{x}_2,\mathfrak{v})(\mathfrak{u}-\mathfrak{v})$ *for* $\mathfrak{u}>\mathfrak{v}\ge 0$, $1\le j,k\le 2$, $\mathfrak{x}_j\in\mathbb{T}_j$, *then*

$$u(\mathfrak{x}_1,\mathfrak{x}_2) \le w^{-1}(\mathfrak{G}_1^{-1}(\mathfrak{G}_2^{-1}(\mathfrak{G}_2(\mathfrak{b}_2(\mathfrak{x}_1,\mathfrak{x}_2))+a_2(\mathfrak{x}_1,\mathfrak{x}_2)\{\mathfrak{c}(\mathfrak{x}_1,\mathfrak{x}_2)$$
$$+\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} r_i(\mathfrak{t}_1,\mathfrak{t}_2)\mathrm{M}(\mathfrak{t}_1,\mathfrak{t}_2,0)\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\}))). \qquad (2.2.43)$$

Proof. Under the condition (**C2**), the inequality (2.2.3) is rewritten as:

$$w(u(\mathfrak{x}_1,\mathfrak{x}_2)) \le a_1(\mathfrak{y},\mathfrak{x}_2)+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))$$
$$\times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_2(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))$$
$$+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)w_2(u(\mu_{1i}(\mathfrak{m}_1),\mu_{2i}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}$$
$$+r_i(\mathfrak{t}_1,\mathfrak{t}_2)\mathrm{L}(\mathfrak{t}_1,\mathfrak{t}_2,w_2(u(\mathfrak{t}_1,\mathfrak{t}_2)))]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1,$$

provided that $(\mathfrak{x}_1,\mathfrak{x}_2)\in[\mathfrak{x}_{01},\mathfrak{y}]_{\mathbb{T}}\times\mathbb{T}_2$ for some fixed $\mathfrak{y}\in\mathbb{T}_1$.
On letting $\xi_3(\mathfrak{x}_1,\mathfrak{x}_2)$ by

$$\xi_3(\mathfrak{x}_1,\mathfrak{x}_2) := a_1(\mathfrak{y},\mathfrak{x}_2)+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))$$
$$\times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_2(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))$$
$$+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)w_2(u(\mu_{1i}(\mathfrak{m}_1),\mu_{2i}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}$$
$$+r_i(\mathfrak{t}_1,\mathfrak{t}_2)\mathrm{L}(\mathfrak{t}_1,\mathfrak{t}_2,w_2(u(\mathfrak{t}_1,\mathfrak{t}_2)))]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1, \qquad (2.2.44)$$

then we have

$$\xi_3(\mathfrak{x}_{01}, \mathfrak{x}_2) = a_1(\mathfrak{y}, \mathfrak{x}_2), \tag{2.2.45}$$

and

$$u(\mathfrak{x}_1, \mathfrak{x}_2) \leq w^{-1}(\xi_3(\mathfrak{x}_1, \mathfrak{x}_2)). \tag{2.2.46}$$

On going the identical steps from (2.2.8) – (2.2.10), we have

$$u(\mu_{1i}(\mathfrak{x}_1), \mu_{2i}(\mathfrak{x}_2)) \leq w^{-1}(\xi_3(\mathfrak{x}_1, \mathfrak{x}_2)) \text{ for } (\mathfrak{x}_1, \mathfrak{x}_2) \in [\mathfrak{x}_{01}, \mathfrak{y}]_{\mathbb{T}} \times \mathbb{T}_2. \tag{2.2.47}$$

From (2.2.44), by [9, Lemma 1.2] we have

$$\begin{aligned}
{\xi_3}^{\Delta \mathfrak{x}_1}(\mathfrak{x}_1, \mathfrak{x}_2) \;=\;& a_2(\mathfrak{y}, \mathfrak{x}_2) \sum_{i=1}^{n} \gamma_{1i}^{\Delta}(\mathfrak{x}_1) \int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(u(\mu_{1i}(\gamma_{1i}(\mathfrak{x}_1)), \mu_{2i}(\mathfrak{t}_2))) \\
&\times [f_i(\sigma(\mathfrak{x}_1), \gamma_{1i}(\mathfrak{x}_1), \mathfrak{x}_2, \mathfrak{t}_2)\{w_2(u(\mu_{1i}(\gamma_{1i}(\mathfrak{x}_1)), \mu_{2i}(\mathfrak{t}_2))) \\
&+ \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)} \int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1), \mathfrak{m}_1, \mathfrak{t}_2, \mathfrak{m}_2) \\
&\times w_2(u(\mu_{1i}(\mathfrak{m}_1), \mu_{2i}(\mathfrak{m}_2)))\Delta \mathfrak{m}_2 \Delta \mathfrak{m}_1\} \\
&+ r_i(\gamma_{1i}(\mathfrak{x}_1), \mathfrak{t}_2)\mathrm{L}(\gamma_{1i}(\mathfrak{x}_1), \mathfrak{t}_2, w_2(u(\gamma_{1i}(\mathfrak{x}_1), \mathfrak{t}_2)))]\Delta \mathfrak{t}_2 \\
&+ a_2(\mathfrak{y}, \mathfrak{x}_2) \sum_{i=1}^{n} \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)} [\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(u(\mu_{1i}(\mathfrak{t}_1), \mu_{2i}(\mathfrak{t}_2))) \\
&\times [f_i(\mathfrak{x}_1, \mathfrak{t}_1, \mathfrak{x}_2, \mathfrak{t}_2)\{w_2(u(\mu_{1i}(\mathfrak{t}_1), \mu_{2i}(\mathfrak{t}_2))) \\
&+ \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1} \int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1, \mathfrak{m}_1, \mathfrak{t}_2, \mathfrak{m}_2) w_2(u(\mu_{1i}(\mathfrak{m}_1), \mu_{2i}(\mathfrak{m}_2)))\Delta \mathfrak{m}_2 \Delta \mathfrak{m}_1\} \\
&+ r_i(\mathfrak{t}_1, \mathfrak{t}_2)\mathrm{L}(\mathfrak{t}_1, \mathfrak{t}_2, w_2(u(\mathfrak{t}_1, \mathfrak{t}_2)))]\Delta \mathfrak{t}_2]^{\Delta \mathfrak{x}_1} \Delta \mathfrak{t}_1.
\end{aligned} \tag{2.2.48}$$

Plugging inequality (2.2.47) in (2.2.48), we have

$$\begin{aligned}
{\xi_3}^{\Delta \mathfrak{x}_1}(\mathfrak{x}_1, \mathfrak{x}_2) \;=\;& a_2(\mathfrak{y}, \mathfrak{x}_2) \sum_{i=1}^{n} \gamma_{1i}^{\Delta}(\mathfrak{x}_1) \int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(w^{-1}(\xi_3(\gamma_{1i}(\mathfrak{x}_1), \mathfrak{t}_2))) \\
&\times [f_i(\sigma(\mathfrak{x}_1), \gamma_{1i}(\mathfrak{x}_1), \mathfrak{x}_2, \mathfrak{t}_2)\{w_2(w^{-1}(\xi_3(\gamma_{1i}(\mathfrak{x}_1), \mathfrak{t}_2))) \\
&+ \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)} \int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1), \mathfrak{m}_1, \mathfrak{t}_2, \mathfrak{m}_2) \\
&\times w_2(w^{-1}(\xi_3(\mathfrak{m}_1, \mathfrak{m}_2)))\Delta \mathfrak{m}_2 \Delta \mathfrak{m}_1\} \\
&+ r_i(\gamma_{1i}(\mathfrak{x}_1), \mathfrak{t}_2)\mathrm{L}(\gamma_{1i}(\mathfrak{x}_1), \mathfrak{t}_2, w_2(w^{-1}(\xi_3(\mathfrak{x}_1, \mathfrak{t}_2))))]\Delta \mathfrak{t}_2 \\
&+ a_2(\mathfrak{y}, \mathfrak{x}_2) \sum_{i=1}^{n} \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)} [\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_1(w^{-1}(\xi_3(\mathfrak{t}_1, \mathfrak{t}_2))) \\
&\times [f_i(\mathfrak{x}_1, \mathfrak{t}_1, \mathfrak{x}_2, \mathfrak{t}_2)\{w_2(w^{-1}(\xi_3(\mathfrak{t}_1, \mathfrak{t}_2))) \\
&+ \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1} \int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1, \mathfrak{m}_1, \mathfrak{t}_2, \mathfrak{m}_2) w_2(w^{-1}(\xi_3(\mathfrak{m}_1, \mathfrak{m}_2)))\Delta \mathfrak{m}_2 \Delta \mathfrak{m}_1\} \\
&+ r_i(\mathfrak{t}_1, \mathfrak{t}_2)\mathrm{L}(\mathfrak{t}_1, \mathfrak{t}_2, w_2(w^{-1}(\xi_3(\mathfrak{t}_1, \mathfrak{t}_2))))]\Delta \mathfrak{t}_2]^{\Delta \mathfrak{x}_1} \Delta \mathfrak{t}_1.
\end{aligned} \tag{2.2.49}$$

Using the fact that w_1, w, ξ_3 are nondecreasing, (2.2.49) become

$$\begin{aligned}
{\xi_3}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \leq\ & w_1(w^{-1}(\xi_3(\mathfrak{x}_1,\mathfrak{x}_2)))a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(w^{-1}(\xi_3(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2}g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(w^{-1}(\xi_3(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)\\
&\times \mathrm{L}(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2,w_2(w^{-1}(\xi_3(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2))))]\Delta\mathfrak{t}_2\\
&+w_1(w^{-1}(\xi_3(\mathfrak{x}_1,\mathfrak{x}_2)))a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(w^{-1}(\xi_3(\mathfrak{t}_1,\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2}g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(w^{-1}(\xi_3(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&+r_i(\mathfrak{t}_1,\mathfrak{t}_2)\mathrm{L}(\mathfrak{t}_1,\mathfrak{t}_2,w_2(w^{-1}(\xi_3(\mathfrak{t}_1,\mathfrak{t}_2))))]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1. \qquad (2.2.50)
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\frac{{\xi_3}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{w_1(w^{-1}(\xi_3(\mathfrak{x}_1,\mathfrak{x}_2)))} \leq\ & a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(w^{-1}(\xi_3(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))\\
&+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2}g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(w^{-1}(\xi_3(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)\\
&\times \mathrm{L}(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2,w_2(w^{-1}(\xi_3(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2))))]\Delta\mathfrak{t}_2\\
&+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(w^{-1}(\xi_3(\mathfrak{t}_1,\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2}g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(w^{-1}(\xi_3(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&+r_i(\mathfrak{t}_1,\mathfrak{t}_2)\mathrm{L}(\mathfrak{t}_1,\mathfrak{t}_2,w_2(w^{-1}(\xi_3(\mathfrak{t}_1,\mathfrak{t}_2))))]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1\\
=\ & a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}[\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(w^{-1}(\xi_3(\mathfrak{t}_1,\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2}g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(w^{-1}(\xi_3(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&+r_i(\mathfrak{t}_1,\mathfrak{t}_2)\mathrm{L}(\mathfrak{t}_1,\mathfrak{t}_2,w_2(w^{-1}(\xi_3(\mathfrak{t}_1,\mathfrak{t}_2))))]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1]^{\Delta\mathfrak{x}_1}.
\end{aligned}$$

Integrating over $[\mathfrak{x}_{01}, \mathfrak{x}_1]$ and using the definition of $\mathfrak{G}_1$ yields:

$$\begin{aligned}
\mathfrak{G}_1(\xi_3(\mathfrak{x}_1,\mathfrak{x}_2)) &\le \mathfrak{G}_1(\xi_3(\mathfrak{x}_{01},\mathfrak{x}_2)) + a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\quad \times\{w_2(w^{-1}(\xi_3(\mathfrak{t}_1,\mathfrak{t}_2))) + \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(t_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\quad \times w_2(w^{-1}(\xi_3(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\} + r_i(\mathfrak{t}_1,\mathfrak{t}_2)\\
&\quad \times \mathrm{L}(\mathfrak{t}_1,\mathfrak{t}_2,w_2(w^{-1}(\xi_3(\mathfrak{t}_1,\mathfrak{t}_2))))]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\\
&\le \mathfrak{G}_1(a_1(\mathfrak{y},\mathfrak{x}_2)) + a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\quad \times\{w_2(w^{-1}(\xi_3(\mathfrak{t}_1,\mathfrak{t}_2))) + \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(t_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\quad \times w_2(w^{-1}(\xi_3(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\} + r_i(\mathfrak{t}_1,\mathfrak{t}_2)\\
&\quad \times\{\mathrm{L}(\mathfrak{t}_1,\mathfrak{t}_2,0) + \mathrm{M}(\mathfrak{t}_1,\mathfrak{t}_2,0)w_2(w^{-1}(\xi_3(\mathfrak{t}_1,\mathfrak{t}_2)))\}]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\\
&\le \mathfrak{G}_1(a_1(\mathfrak{y},\mathfrak{x}_2)) + a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{y})}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} r_i(\mathfrak{t}_1,\mathfrak{t}_2)\\
&\quad \times \mathrm{L}(\mathfrak{t}_1,\mathfrak{t}_2,0)\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\\
&\quad + a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_2(w^{-1}(\xi_3(\mathfrak{t}_1,\mathfrak{t}_2)))[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\quad \times\{1 + \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&\quad + r_i(\mathfrak{t}_1,\mathfrak{t}_2)\mathrm{M}(\mathfrak{t}_1,\mathfrak{t}_2,0)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\\
&= \mathfrak{b}_2(\mathfrak{y},\mathfrak{x}_2) + a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_2(w^{-1}(\xi_3(\mathfrak{t}_1,\mathfrak{t}_2)))\\
&\quad \times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{1 + \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&\quad + r_i(\mathfrak{t}_1,\mathfrak{t}_2)\mathrm{M}(\mathfrak{t}_1,\mathfrak{t}_2,0)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1.
\end{aligned}$$

On letting $\zeta_3(\mathfrak{x}_1,\mathfrak{x}_2)$ by

$$\begin{aligned}
\zeta_3(\mathfrak{x}_1,\mathfrak{x}_2) &:= \mathfrak{b}_2(\mathfrak{y},\mathfrak{x}_2) + a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_2(w^{-1}(\xi_3(\mathfrak{t}_1,\mathfrak{t}_2)))\\
&\quad \times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{1 + \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&\quad + r_i(\mathfrak{t}_1,\mathfrak{t}_2)\mathrm{M}(\mathfrak{t}_1,\mathfrak{t}_2,0)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1, \qquad (2.2.51)
\end{aligned}$$

then we have

$$\zeta_3(\mathfrak{x}_{01},\mathfrak{x}_2) = \mathfrak{b}_2(\mathfrak{y},\mathfrak{x}_2), \qquad (2.2.52)$$

and

$$\xi_3(\mathfrak{x}_1,\mathfrak{x}_2) \le \mathfrak{G}_1^{-1}\left(\zeta_3(\mathfrak{x}_1,\mathfrak{x}_2)\right). \qquad (2.2.53)$$

$$\begin{aligned}
\Rightarrow {\zeta_3}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \quad = \quad & a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_2(w^{-1}(\xi_3(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))\\
&\times[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&+r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)\mathrm{M}(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2,0)]\Delta\mathfrak{t}_2\\
&+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} w_2(w^{-1}(\xi_3(\mathfrak{t}_1,\mathfrak{t}_2)))\\
&\times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&+r_i(\mathfrak{t}_1,\mathfrak{t}_2)\mathrm{M}(\mathfrak{t}_1,\mathfrak{t}_2,0)]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1. \qquad (2.2.54)
\end{aligned}$$

Plugging inequality (2.2.53) in (2.2.54), then using the fact that w_2, w, $\mathfrak{G}_1$, ζ_3, are nondecreasing, (2.2.54) become

$$\begin{aligned}
{\zeta_3}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \quad \leq \quad & w_2(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_3(\mathfrak{x}_1,\mathfrak{x}_2))))a_2(\mathfrak{y},\mathfrak{x}_2)\\
&\times\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&+r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)\mathrm{M}(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2,0)]\Delta\mathfrak{t}_2\\
&+w_2(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_3(\mathfrak{x}_1,\mathfrak{x}_2))))a_2(\mathfrak{y},\mathfrak{x}_2)\\
&\times\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&+r_i(\mathfrak{t}_1,\mathfrak{t}_2)\mathrm{M}(\mathfrak{t}_1,\mathfrak{t}_2,0)]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\frac{{\zeta_3}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{w_2(w^{-1}(\mathfrak{G}_1^{-1}(\zeta_3(\mathfrak{x}_1,\mathfrak{x}_2))))} \quad \leq \quad & a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&+r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)\mathrm{M}(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2,0)]\Delta\mathfrak{t}_2\\
&+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\
&+r_i(\mathfrak{t}_1,\mathfrak{t}_2)\mathrm{M}(\mathfrak{t}_1,\mathfrak{t}_2,0)]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1
\end{aligned}$$

$$= a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}[\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)$$
$$\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}$$
$$+r_i(\mathfrak{t}_1,\mathfrak{t}_2)\mathrm{M}(\mathfrak{t}_1,\mathfrak{t}_2,0)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1]^{\Delta\mathfrak{x}_1}.$$

Integrating over $[\mathfrak{x}_{01},\mathfrak{x}_1]$ and using the definition of $\mathfrak{G}_2$ yields:

$$\begin{aligned}\mathfrak{G}_2(\zeta_3(\mathfrak{x}_1,\mathfrak{x}_2)) &\le \mathfrak{G}_2(\zeta_3(\mathfrak{x}_{01},\mathfrak{x}_2)) + a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\ &\quad\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\\ &\quad+r_i(\mathfrak{t}_1,\mathfrak{t}_2)\mathrm{M}(\mathfrak{t}_1,\mathfrak{t}_2,0)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\\ &= \mathfrak{G}_2(\mathfrak{b}_2(\mathfrak{y},\mathfrak{x}_2)) + a_2(\mathfrak{y},\mathfrak{x}_2)\{\mathfrak{c}(\mathfrak{x}_1,\mathfrak{x}_2)\\ &\quad+\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} r_i(\mathfrak{t}_1,\mathfrak{t}_2)\mathrm{M}(\mathfrak{t}_1,\mathfrak{t}_2,0)\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\} \qquad (2.2.55)\end{aligned}$$

Combination of (2.2.46), (2.2.53) and (2.2.55) yield the desired result (2.2.43). □

Remark 2.2.3 *For* $\mathbb{T}=\mathbf{R}$, $w_2\equiv I\equiv\mu_{ji}$, $f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)=f_i(\mathfrak{t}_1,\mathfrak{t}_2)$, $g_i\equiv 0$, *Theorems 2.2.3 coincide with [28, Theorem 3].*

Theorem 2.2.4 *Let the inequalities* (2.2.1) *and* (2.2.4) *be hold and under the conditions* (**C1**) – (**C6**) *for* $\mathfrak{x}_j\in\mathbb{T}_j$, $1\le j,k\le 2$ *such that* $w_2\left(w^{-1}\left(\mathfrak{G}_1^{-1}(\mathfrak{u})\right)\right)\le\mathfrak{u}$, *one has*

$$u(\mathfrak{x}_1,\mathfrak{x}_2)\le w^{-1}(\mathfrak{G}_1^{-1}(\mathfrak{b}_1(\mathfrak{x}_1,\mathfrak{x}_2)\exp_{a_2(\mathfrak{x}_1,\mathfrak{x}_2)\mathfrak{c}^{\Delta}(.,\mathfrak{x}_2)}(\mathfrak{x}_1,\mathfrak{x}_{01}))). \qquad (2.2.56)$$

Proof. On going the identical steps from (2.2.6)-(2.2.12) and from (2.2.12), we have

$$\begin{aligned}\mathfrak{G}_1(\xi_1(\mathfrak{x}_1,\mathfrak{x}_2)) &\le \mathfrak{b}_1(\mathfrak{x}_1,\mathfrak{x}_2)+a_2(\mathfrak{x}_1,\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\ &\quad\times\{w_2(w^{-1}(\xi_1(\mathfrak{t}_1,\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ &\quad\times w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\\ \Rightarrow\frac{\mathfrak{G}_1(\xi_1(\mathfrak{x}_1,\mathfrak{x}_2))}{\mathfrak{b}_1(\mathfrak{x}_1,\mathfrak{x}_2)+\epsilon} &\le 1+a_2(\mathfrak{x}_1,\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\ &\quad\times\{\frac{w_2(w^{-1}(\xi_1(\mathfrak{t}_1,\mathfrak{t}_2)))}{\mathfrak{b}_1(\mathfrak{t}_1,\mathfrak{t}_2)+\epsilon}+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ &\quad\times\frac{w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))}{\mathfrak{b}_1(\mathfrak{m}_1,\mathfrak{m}_2)+\epsilon}\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1 \qquad (2.2.57)\end{aligned}$$

On letting $\zeta_4(\mathfrak{x}_1,\mathfrak{x}_2)$ by

$$\begin{aligned}\zeta_4(\mathfrak{x}_1,\mathfrak{x}_2) &:= 1+a_2(\mathfrak{x}_1,\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\ &\times\{\frac{w_2(w^{-1}(\xi_1(\mathfrak{t}_1,\mathfrak{t}_2)))}{\mathfrak{b}_1(\mathfrak{t}_1,\mathfrak{t}_2)+\epsilon}+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ &\times\frac{w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))}{\mathfrak{b}_1(\mathfrak{m}_1,\mathfrak{m}_2)+\epsilon}\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1, \qquad (2.2.58)\end{aligned}$$

then we have

$$\xi_1(\mathfrak{x}_1,\mathfrak{x}_2)\le \mathfrak{G}_1^{-1}((\mathfrak{b}_1(\mathfrak{x}_1,\mathfrak{x}_2)+\epsilon)\zeta_4(\mathfrak{x}_1,\mathfrak{x}_2)) \qquad (2.2.59)$$

From (2.2.58), by [9, Lemma 1.2] we have

$$\begin{aligned}\zeta_4{}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) &= a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\ &\times\{\frac{w_2(w^{-1}(\xi_1(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)))}{\mathfrak{b}_1(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)+\epsilon}+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ &\times\frac{w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))}{\mathfrak{b}_1(\mathfrak{m}_1,\mathfrak{m}_2)+\epsilon}\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\\ &+a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\ &\times\{\frac{w_2(w^{-1}(\xi_1(\mathfrak{t}_1,\mathfrak{t}_2)))}{\mathfrak{b}_1(\mathfrak{t}_1,\mathfrak{t}_2)+\epsilon}+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ &\times\frac{w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))}{\mathfrak{b}_1(\mathfrak{m}_1,\mathfrak{m}_2)+\epsilon}\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1. \qquad (2.2.60)\end{aligned}$$

Since

$$\begin{aligned}\mathfrak{G}_1(\xi_1(\mathfrak{x}_1,\mathfrak{x}_2)) &\le \mathfrak{b}_1(\mathfrak{x}_1,\mathfrak{x}_2)+a_2(\mathfrak{x}_1,\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\ &\times\{w_2(w^{-1}(\xi_1(\mathfrak{t}_1,\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ &\times w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\\ \Rightarrow w_2(w^{-1}(\xi_1(\mathfrak{x}_1,\mathfrak{x}_2))) &\le \mathfrak{b}_1(\mathfrak{x}_1,\mathfrak{x}_2)+a_2(\mathfrak{x}_1,\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\ &\times\{w_2(w^{-1}(\xi_1(\mathfrak{t}_1,\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ &\times w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\\ \Rightarrow \frac{w_2(w^{-1}(\xi_1(\mathfrak{x}_1,\mathfrak{x}_2)))}{\mathfrak{b}_1(\mathfrak{y},\mathfrak{x}_2)+\epsilon} &\le 1+a_2(\mathfrak{x}_1,\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\ &\times\{\frac{w_2(w^{-1}(\xi_1(\mathfrak{t}_1,\mathfrak{t}_2)))}{\mathfrak{b}_1(\mathfrak{t}_1,\mathfrak{t}_2)+\epsilon}+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\end{aligned}$$

$$\times\frac{w_2(w^{-1}(\xi_1(\mathfrak{m}_1,\mathfrak{m}_2)))}{\mathfrak{b}_1(\mathfrak{m}_1,\mathfrak{m}_2)+\epsilon}\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1$$
$$= \zeta_4(\mathfrak{x}_1,\mathfrak{x}_2) \tag{2.2.61}$$

From (2.2.60) and (2.2.61), we have

$$\begin{aligned}
{\zeta_4}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) &\leq \zeta_4(\mathfrak{x}_1,\mathfrak{x}_2)a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times(1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1)\Delta\mathfrak{t}_2\\
&+\zeta_4(\mathfrak{x}_1,\mathfrak{x}_2)a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times(1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1)\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1\\
&= \zeta_4(\mathfrak{x}_1,\mathfrak{x}_2)a_2(\mathfrak{y},\mathfrak{x}_2)\sum_{i=1}^{n}[\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times(1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1)\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1]^{\Delta\mathfrak{x}_1}\\
&= \zeta_4(\mathfrak{x}_1,\mathfrak{x}_2)a_2(\mathfrak{y},\mathfrak{x}_2)\mathfrak{c}^{\Delta}(\mathfrak{x}_1,\mathfrak{x}_2).
\end{aligned}$$

By Lemma 1.5.5 we have

$$\zeta_4(\mathfrak{x}_1,\mathfrak{x}_2) \leq \exp_{a_2(\mathfrak{y},\mathfrak{x}_2)\mathfrak{c}^{\Delta}(.,\mathfrak{x}_2)}(\mathfrak{x}_1,\mathfrak{x}_{01}) \tag{2.2.62}$$

Combination of (2.2.8), (2.2.59) and (2.2.62) yields:

$$u(\mathfrak{x}_1,\mathfrak{x}_2) \leq w^{-1}(\mathfrak{G}_1^{-1}((\mathfrak{b}_1(\mathfrak{x}_1,\mathfrak{x}_2)+\epsilon)\exp_{a_2(\mathfrak{y},\mathfrak{x}_2)\mathfrak{c}^{\Delta}(.,\mathfrak{x}_2)}(\mathfrak{x}_1,\mathfrak{x}_{01}))) \tag{2.2.63}$$

letting $\epsilon \to 0$, yields the desired result (2.2.56). □

Corollary 2.2.5 [*23*] *Let the conditions* (**C1**) − (**C6**) *for* $1 \leq k \leq 2$ *be satisfied, if* $q_1 > q_2 > 0$ *and* $\mathfrak{C} \geq 0$ *are constants such that:*

$$\begin{aligned}
u^{q_1}(\mathfrak{x}_1,\mathfrak{x}_2) &\leq \mathfrak{C}+\frac{q_1}{q_1-q_2}\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} u^{q_2}(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2))[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(u(\mu_{1i}(\mathfrak{m}_1),\mu_{2i}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1
\end{aligned} \tag{2.2.64}$$

for $(\mathfrak{x}_1,\mathfrak{x}_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ *with the initial condition*

$$\begin{cases} u(\mathfrak{x}_1,\mathfrak{x}_2) = \overline{\mathfrak{a}}(\mathfrak{x}_1,\mathfrak{x}_2), & \mathfrak{x}_1 \in [\mathfrak{p}_1,\mathfrak{x}_{01}]_{\mathbb{T}} \text{ or } \mathfrak{x}_2 \in [\mathfrak{p}_2,\mathfrak{x}_{02}]_{\mathbb{T}}\ ;\\ \overline{\mathfrak{a}}(\mu_{1i}(\mathfrak{x}_1),\mu_{2i}(\mathfrak{x}_2)) \leq \sqrt[q_1]{\mathfrak{C}}, & \mu_{1i}(\mathfrak{x}_1) \leq \mathfrak{x}_{01} \text{ or } \mu_{2i}(\mathfrak{x}_2) \leq \mathfrak{x}_{02}, \end{cases} \tag{2.2.65}$$

then

$$u(\mathfrak{x}_1,\mathfrak{x}_2) \leq \sqrt[q_1-q_2]{\mathfrak{H}_1{}^{-1}(\mathfrak{H}_1(\overline{\mathfrak{b}}_1(\mathfrak{x}_1,\mathfrak{x}_2)) + \mathfrak{c}(\mathfrak{x}_1,\mathfrak{x}_2))} \quad \textit{for } (\mathfrak{x}_1,\mathfrak{x}_2) \in \mathbb{T}_1 \times \mathbb{T}_2, \tag{2.2.66}$$

provided that:

$$\mathfrak{H}_1(\mathfrak{r}) := \int_{\mathfrak{x}_{01}+1}^{\mathfrak{r}} \frac{\Delta p}{w_2(\sqrt[q_1-q_2]{p})}, \qquad \textit{for } \overline{\mathfrak{H}}_1(\infty) = \infty,$$

$$\overline{\mathfrak{b}}_1(\mathfrak{x}_1,\mathfrak{x}_2) := \mathfrak{C}^{\frac{q_1-q_2}{q_1}} + \sum_{i=1}^{n} \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)} \int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} r_i(\mathfrak{t}_1,\mathfrak{t}_2)\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1.$$

Proof. On letting $\overline{\xi}_1(\mathfrak{x}_1,\mathfrak{x}_2)$ by

$$\begin{aligned}\overline{\xi}_1(\mathfrak{x}_1,\mathfrak{x}_2) \;:=\; & \mathfrak{C} + \frac{q_1}{q_1-q_2}\sum_{i=1}^{n} \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)} \int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} u^{q_2}(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2))[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\ & \times\{w_2(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2))) + \int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1} \int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\ & \times w_2(u(\mu_{1i}(\mathfrak{m}_1),\mu_{2i}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\} + r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1, \end{aligned} \tag{2.2.67}$$

then we have

$$\overline{\xi}_1(\mathfrak{x}_{01},\mathfrak{x}_2) = \mathfrak{C}, \tag{2.2.68}$$

and

$$u(\mathfrak{x}_1,\mathfrak{x}_2) \leq \sqrt[q_1]{\overline{\xi}_1(\mathfrak{x}_1,\mathfrak{x}_2)}. \tag{2.2.69}$$

If $\mu_{1i}(\mathfrak{x}_1) \geq \mathfrak{x}_{01}$ and $\mu_{2i}(\mathfrak{x}_2) \geq \mathfrak{x}_{02}$, then

$$u(\mu_{1i}(\mathfrak{x}_1),\mu_{2i}(\mathfrak{x}_2)) \leq \sqrt[q_1]{\overline{\xi}_1(\mu_{1i}(\mathfrak{x}_1),\mu_{2i}(\mathfrak{x}_2))} \leq \sqrt[q_1]{\overline{\xi}_1(\mathfrak{x}_1,\mathfrak{x}_2)}. \tag{2.2.70}$$

If $\mu_{1i}(\mathfrak{x}_1) \leq \mathfrak{x}_{01}$ or $\mu_{2i}(\mathfrak{x}_2) \leq \mathfrak{x}_{02}$, then

$$\begin{aligned} u(\mu_{1i}(\mathfrak{x}_1),\mu_{2i}(\mathfrak{x}_2)) &= \overline{\mathfrak{a}}(\mu_{1i}(\mathfrak{x}_1),\mu_{2i}(\mathfrak{x}_2))\\ &\leq \sqrt[q_1]{\mathfrak{C}} \leq \sqrt[q_1]{\overline{\xi}_1(\mathfrak{x}_1,\mathfrak{x}_2)}. \end{aligned} \tag{2.2.71}$$

From (2.2.70) and (2.2.71), we have

$$u(\mu_{1i}(\mathfrak{x}_1),\mu_{2i}(\mathfrak{x}_2)) \leq \sqrt[q_1]{\overline{\xi}_1(\mathfrak{x}_1,\mathfrak{x}_2)}, \tag{2.2.72}$$

and hence

$$\begin{aligned}\overline{\xi}_1{}^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \;=\; & \frac{q_1}{q_1-q_2}\sum_{i=1}^{n} \gamma_{1i}^{\Delta}(\mathfrak{x}_1) \int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} u^{q_2}(\mu_{1i}(\gamma_{1i}(\mathfrak{x}_1)),\mu_{2i}(\mathfrak{t}_2))\\ & \times[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\{w_2(u(\mu_{1i}(\gamma_{1i}(\mathfrak{x}_1)),\mu_{2i}(\mathfrak{t}_2)))\end{aligned}$$

$$
\begin{aligned}
&+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(u(\mu_{1i}(\mathfrak{m}_1),\mu_{2i}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\} + r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)]\Delta\mathfrak{t}_2\\
&+\frac{q_1}{q_1-q_2}\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} u^{q_2}(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2))[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2)))+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(u(\mu_{1i}(\mathfrak{m}_1),\mu_{2i}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\} + r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1.
\end{aligned}
\tag{2.2.73}
$$

Plugging inequality (2.2.72) in (2.2.73), then using the fact that $\overline{\xi}_1$ is nondecreasing, the equation (2.2.73) become

$$
\begin{aligned}
\overline{\xi}_1^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \le\ & \frac{q_1}{q_1-q_2}\overline{\xi}_1^{\frac{q_2}{q_1}}(\mathfrak{x}_1,\mathfrak{x}_2)\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(\sqrt[q_1]{\overline{\xi}_1(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)})+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{m}_1,\mathfrak{m}_2)})\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\} + r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)]\Delta\mathfrak{t}_2\\
&+\frac{q_1}{q_1-q_2}\overline{\xi}_1^{\frac{q_2}{q_1}}(\mathfrak{x}_1,\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{t}_1,\mathfrak{t}_2)})+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{m}_1,\mathfrak{m}_2)})\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\} + r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1.
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
\frac{\overline{\xi}_1^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{\overline{\xi}_1^{\frac{q_2}{q_1}}(\mathfrak{x}_1,\mathfrak{x}_2)} \le\ & \frac{q_1}{q_1-q_2}\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(\sqrt[q_1]{\overline{\xi}_1(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)})+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{m}_1,\mathfrak{m}_2)})\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\} + r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)]\Delta\mathfrak{t}_2\\
&+\frac{q_1}{q_1-q_2}\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{t}_1,\mathfrak{t}_2)})+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{m}_1,\mathfrak{m}_2)})\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\} + r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1.
\end{aligned}
\tag{2.2.74}
$$

By Theorem 1.2.9, we have

$$
\left(\frac{q_1}{q_1-q_2}\overline{\xi}_1^{\frac{q_1-q_2}{q_1}}(\mathfrak{x}_1,\mathfrak{x}_2)\right)^{\Delta\mathfrak{x}_1}
$$

$$= \overline{\xi}_1^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)\int_0^1\{\overline{\xi}_1(\mathfrak{x}_1,\mathfrak{x}_2)+h\mu(\mathfrak{x}_1,\mathfrak{x}_2)\overline{\xi}_1^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)\}^{-\frac{q_2}{q_1}}dh$$

$$= \frac{\overline{\xi}_1^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{\overline{\xi}_1^{\frac{q_2}{q_1}}(\mathfrak{x}_1,\mathfrak{x}_2)}\int_0^1\{1+h\mu(\mathfrak{x}_1,\mathfrak{x}_2)\frac{\overline{\xi}_1^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{\overline{\xi}_1(\mathfrak{x}_1,\mathfrak{x}_2)}\}^{-\frac{q_2}{q_1}}dh$$

$$= \frac{\overline{\xi}_1^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{\overline{\xi}_1^{\frac{q_2}{q_1}}(\mathfrak{x}_1,\mathfrak{x}_2)}\times\left|\frac{\{1+h\mu(\mathfrak{x}_1,\mathfrak{x}_2)\frac{\overline{\xi}_1^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{\overline{\xi}_1(\mathfrak{x}_1,\mathfrak{x}_2)}\}^{-\frac{q_2}{q_1}+1}}{\mu(\mathfrak{x}_1,\mathfrak{x}_2)\frac{\overline{\xi}_1^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{\overline{\xi}_1(\mathfrak{x}_1,\mathfrak{x}_2)}(1-\frac{q_2}{q_1})}\right|_0^1$$

$$= \frac{\overline{\xi}_1^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{\overline{\xi}_1^{\frac{q_2}{q_1}}(\mathfrak{x}_1,\mathfrak{x}_2)}\frac{\{1+\mu(\mathfrak{x}_1,\mathfrak{x}_2)\frac{\overline{\xi}_1^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{\overline{\xi}_1(\mathfrak{x}_1,\mathfrak{x}_2)}\}^{-\frac{q_2}{q_1}+1}-1}{\mu(\mathfrak{x}_1,\mathfrak{x}_2)\frac{\overline{\xi}_1^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{\overline{\xi}_1(\mathfrak{x}_1,\mathfrak{x}_2)}(1-\frac{q_2}{q_1})} \tag{2.2.75}$$

By Theorem 1.5.6 for $\overline{\xi}_1^{\Delta\mathfrak{x}_1}\geq 0$, we have

$$(\frac{q_1}{q_1-q_2}\overline{\xi}_1^{\frac{q_1-q_2}{q_1}}(\mathfrak{x}_1,\mathfrak{x}_2))^{\Delta\mathfrak{x}_1} \leq \frac{\overline{\xi}_1^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{\overline{\xi}_1^{\frac{q_2}{q_1}}(\mathfrak{x}_1,\mathfrak{x}_2)} \tag{2.2.76}$$

From (2.2.74) and (2.2.76), we have

$$\begin{aligned}
(\frac{q_1}{q_1-q_2}\overline{\xi}_1^{\frac{q_1-q_2}{q_1}}(\mathfrak{x}_1,\mathfrak{x}_2))^{\Delta\mathfrak{x}_1} \leq\ & \frac{q_1}{q_1-q_2}\sum_{i=1}^n\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(\sqrt[q_1]{\overline{\xi}_1(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)})\\
&+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2}g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{m}_1,\mathfrak{m}_2)})\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)]\Delta\mathfrak{t}_2\\
&+\frac{q_1}{q_1-q_2}\sum_{i=1}^n\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{t}_1,\mathfrak{t}_2)})+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2}g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{m}_1,\mathfrak{m}_2)})\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1\\
=\ & \frac{q_1}{q_1-q_2}\sum_{i=1}^n[\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{t}_1,\mathfrak{t}_2)})+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2}g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{m}_1,\mathfrak{m}_2)})\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1]^{\Delta\mathfrak{x}_1}.
\end{aligned}$$

Integrating over $[\mathfrak{x}_{01},\mathfrak{x}_1]$, we obtain

$$\begin{aligned}
\frac{q_1}{q_1-q_2}\overline{\xi}_1^{\frac{q_1-q_2}{q_1}}(\mathfrak{x}_1,\mathfrak{x}_2) \leq\ & \frac{q_1}{q_1-q_2}\overline{\xi}_1^{\frac{q_1-q_2}{q_1}}(\mathfrak{x}_{01},\mathfrak{x}_2)+\frac{q_1}{q_1-q_2}\\
&\times\sum_{i=1}^n\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{y})}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)}r_i(\mathfrak{t}_1,\mathfrak{t}_2)\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1
\end{aligned}$$

$$+\frac{q_1}{q_1-q_2}\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)$$
$$\times\{w_2(\sqrt[q_1]{\overline{\xi}_1(t_1,\mathfrak{t}_2)})+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)$$
$$\times w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{m}_1,\mathfrak{m}_2)})\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1$$

$$\begin{aligned}\Rightarrow \overline{\xi}_1^{\frac{q_1-q_2}{q_1}}(\mathfrak{x}_1,\mathfrak{x}_2) \;\le\; & \mathfrak{C}^{\frac{q_1-q_2}{q_1}}+\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{y})}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} r_i(\mathfrak{t}_1,\mathfrak{t}_2)\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\\
&+\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(\sqrt[q_1]{\overline{\xi}_1(t_1,\mathfrak{t}_2)})+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{m}_1,\mathfrak{m}_2)})\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\\
=\; & \overline{\mathfrak{b}}_1(\mathfrak{y},\mathfrak{x}_2)+\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{t}_1,\mathfrak{t}_2)})+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{m}_1,\mathfrak{m}_2)})\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1 \qquad (2.2.77)\end{aligned}$$

On letting $\overline{\zeta}_1(\mathfrak{x}_1,\mathfrak{x}_2)$ by

$$\begin{aligned}\overline{\zeta}_1(\mathfrak{x}_1,\mathfrak{x}_2) \;:=\; & \overline{\mathfrak{b}}_1(\mathfrak{y},\mathfrak{x}_2)+\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{t}_1,\mathfrak{t}_2)})+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{m}_1,\mathfrak{m}_2)})\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1, \qquad (2.2.78)\end{aligned}$$

then we have

$$\overline{\zeta}_1(\mathfrak{x}_{01},\mathfrak{x}_2)=\overline{\mathfrak{b}}_1(\mathfrak{y},\mathfrak{x}_2), \qquad (2.2.79)$$

and

$$\overline{\xi}_1(\mathfrak{x}_1,\mathfrak{x}_2)\le\overline{\zeta}_1^{\frac{q_1}{q_1-q_2}}(\mathfrak{x}_1,\mathfrak{x}_2). \qquad (2.2.80)$$

From (2.2.78), by [9, Lemma 1.2] we have

$$\begin{aligned}\overline{\zeta}_1^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \;=\; & \sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\{w_2(\sqrt[q_1]{\overline{\xi}_1(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{t}_2)})\\
&+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\end{aligned}$$

$$\begin{aligned}
&\times w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{m}_1,\mathfrak{m}_2)})\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\\
&+\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{t}_1,\mathfrak{t}_2)})+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(\sqrt[q_1]{\overline{\xi}_1(\mathfrak{m}_1,\mathfrak{m}_2)})\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1. \qquad (2.2.81)
\end{aligned}$$

Plugging inequality (2.2.80) in (2.2.81), then using the fact that w_2, $\overline{\zeta}_1$ are nondecreasing, the equation (2.2.81) become

$$\begin{aligned}
\overline{\zeta}_1^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \leq\;& w_2(\sqrt[q_1-q_2]{\overline{\zeta}_1(\mathfrak{x}_1,\mathfrak{x}_2)})\sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\\
&+w_2(\sqrt[q_1-q_2]{\overline{\zeta}_1(\mathfrak{x}_1,\mathfrak{x}_2)})\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\frac{\overline{\zeta}_1^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{w_2(\sqrt[q_1-q_2]{\overline{\zeta}_1(\mathfrak{x}_1,\mathfrak{x}_2)})} \leq\;& \sum_{i=1}^{n}\gamma_{1i}^{\Delta}(\mathfrak{x}_1)\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\sigma(\mathfrak{x}_1),\gamma_{1i}(\mathfrak{x}_1),\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\gamma_{1i}(\mathfrak{x}_1),\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\\
&+\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}[\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2]^{\Delta\mathfrak{x}_1}\Delta\mathfrak{t}_1\\
=\;& \sum_{i=1}^{n}[\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1]^{\Delta\mathfrak{x}_1}.
\end{aligned}$$

Integrating over $[\mathfrak{x}_{01},\mathfrak{x}_1]$ and using the definition of $\mathfrak{H}_1$

$$\begin{aligned}
\mathfrak{H}_1\left(\overline{\zeta}_1(\mathfrak{x}_1,\mathfrak{x}_2)\right) \leq\;& \mathfrak{H}_1\left(\overline{\zeta}_1(\mathfrak{x}_{01},\mathfrak{x}_2)\right)+\sum_{i=1}^{n}\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\gamma_{1i}(\mathfrak{x}_1)}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\gamma_{2i}(\mathfrak{x}_2)} f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\left\{1+\int_{\gamma_{1i}(\mathfrak{x}_{01})}^{\mathfrak{t}_1}\int_{\gamma_{2i}(\mathfrak{x}_{02})}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\right\}\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\\
=\;& \mathfrak{H}_1\left(\overline{\mathfrak{b}}_1(\mathfrak{y},\mathfrak{x}_2)\right)+\mathfrak{c}(\mathfrak{x}_1,\mathfrak{x}_2) \qquad (2.2.82)
\end{aligned}$$

Combination of (2.2.69), (2.2.80) and (2.2.82) yield the desired result (2.2.66). □

Letting $\mathbb{T} = \mathbf{Z}$, form Theorem 2.2.1, we easily establish the following result.

Corollary 2.2.6 *[23] Let $u, r_i, a_j : A_1 \times A_2 \to \mathrm{R}_0^+$ and $f_i, g_i, \Delta\mathfrak{x}_1 f_i : A_1^2 \times A_2^2 \to \mathrm{R}_0^+$ be nonnegative real valued functions defined on $A_1 \times A_2$ and $A_1^2 \times A_2^2$ respectively, with a_j is nondecreasing in each variable; let $\gamma_{ji} : A_j \to \mathrm{R}_0^+$ be nonnegative and nondecreasing function defined on A_j with $\gamma_{ji}(\mathfrak{x}_j) \le \mathfrak{x}_j$; let $\tilde{\mathfrak{a}} : \{-\rho_{1i}, \cdots, -1, 0\} \times \{-\rho_{2i}, \cdots, -1, 0\} \to \mathrm{R}_0^+$ be nonnegative function defined on $\{-\rho_{1i}, \cdots, -1, 0\} \times \{-\rho_{2i}, \cdots, -1, 0\}$ and $-\infty < \tilde{\mathfrak{p}}_j = \inf\{\min(\mathfrak{x}_j - \rho_{ji}), \mathfrak{x}_j \in A_j\} \le 0$; let w and w_j are as defined in Theorem 2.2.1.*

If $u(\mathfrak{x}_1, \mathfrak{x}_2)$ satisfies the following discrete inequality

$$\begin{aligned} w(u(\mathfrak{x}_1, \mathfrak{x}_2)) \le & a_1(\mathfrak{x}_1, \mathfrak{x}_2) + a_2(\mathfrak{x}_1, \mathfrak{x}_2) \sum_{i=1}^{n} \sum_{\mathfrak{t}_1=\gamma_{1i}(0)}^{\gamma_{1i}(\mathfrak{x}_1)-1} \sum_{\mathfrak{t}_2=\gamma_{2i}(0)}^{\gamma_{2i}(\mathfrak{x}_2)-1} w_1(u(\mathfrak{t}_1 - \rho_{1i}, \mathfrak{t}_2 - \rho_{2i})) \\ & \times [f_i(\mathfrak{x}_1, \mathfrak{t}_1, \mathfrak{x}_2, \mathfrak{t}_2)\{w_2(u(\mathfrak{t}_1 - \rho_{1i}, \mathfrak{t}_2 - \rho_{2i})) + \prod_{l=1}^{2} \sum_{\mathfrak{m}_l=\gamma_{li}(0)}^{\mathfrak{t}_l-1} g_i(\mathfrak{t}_1, \mathfrak{m}_1, \mathfrak{t}_2, \mathfrak{m}_2) \\ & \times w_2(u(\mathfrak{m}_1 - \rho_{1i}, \mathfrak{m}_2 - \rho_{2i}))\} + r_i(\mathfrak{t}_1, \mathfrak{t}_2)] \end{aligned} \tag{2.2.83}$$

with the following initial condition

$$\begin{cases} w(u(\mathfrak{x}_1, \mathfrak{x}_2)) = \tilde{\mathfrak{a}}(\mathfrak{x}_1, \mathfrak{x}_2), & \mathfrak{x}_1 \in [\tilde{\mathfrak{p}}_1, 0] \text{ or } \mathfrak{x}_2 \in [\tilde{\mathfrak{p}}_2, 0]\ ; \\ \tilde{\mathfrak{a}}(\mathfrak{x}_1 - \rho_{1i}, \mathfrak{x}_2 - \rho_{2i}) \le a_1(\mathfrak{x}_1, \mathfrak{x}_2), & \mathfrak{x}_1 \le \rho_{1i}, \text{ or } \mathfrak{x}_2 \le \rho_{2i}, \end{cases} \tag{2.2.84}$$

then

$$u(\mathfrak{x}_1, \mathfrak{x}_2) \le w^{-1}(\mathfrak{H}_2^{-1}(\mathfrak{H}_3^{-1}(\mathfrak{H}_3(\tilde{\mathfrak{b}}_1(\mathfrak{x}_1, \mathfrak{x}_2)) + a_2(\mathfrak{x}_1, \mathfrak{x}_2)\tilde{\mathfrak{c}}(\mathfrak{x}_1, \mathfrak{x}_2)))), \tag{2.2.85}$$

provided that

$$\tilde{\mathfrak{c}}(\mathfrak{x}_1, \mathfrak{x}_2) := \sum_{i=1}^{n} \sum_{\mathfrak{t}_1=\gamma_{1i}(0)}^{\gamma_{1i}(\mathfrak{x}_1)-1} \sum_{\mathfrak{t}_2=\gamma_{2i}(0)}^{\gamma_{2i}(\mathfrak{x}_2)-1} f_i(\mathfrak{x}_1, \mathfrak{t}_1, \mathfrak{x}_2, \mathfrak{t}_2)(1 + \sum_{\mathfrak{m}_1=\gamma_{1i}(0)}^{\mathfrak{t}_1-1} \sum_{\mathfrak{m}_2=\gamma_{2i}(0)}^{\mathfrak{t}_2-1} g_i(\mathfrak{t}_1, \mathfrak{m}_1, \mathfrak{t}_2, \mathfrak{m}_2)).$$

$$\tilde{\mathfrak{b}}_1(\mathfrak{x}_1, \mathfrak{x}_2) := \mathfrak{H}_2(a_1(\mathfrak{x}_1, \mathfrak{x}_2)) + a_2(\mathfrak{x}_1, \mathfrak{x}_2) \sum_{i=1}^{n} \sum_{\mathfrak{t}_1=\gamma_{1i}(0)}^{\gamma_{1i}(\mathfrak{x}_1)-1} \sum_{\mathfrak{t}_2=\gamma_{2i}(0)}^{\gamma_{2i}(\mathfrak{x}_2)-1} r_i(\mathfrak{t}_1, \mathfrak{t}_2).$$

$$\mathfrak{H}_2(\mathfrak{r}) := \int_1^{\mathfrak{r}} \frac{dp}{w_1(w^{-1}(p))} \quad \text{for } \mathfrak{H}_2(\infty) = \infty.$$

$$\mathfrak{H}_3(\mathfrak{r}) := \int_1^{\mathfrak{r}} \frac{dp}{w_2(w^{-1}(\mathfrak{H}_2^{-1}(p)))} \quad \text{for } \mathfrak{H}_3(\infty) = \infty.$$

2.3 Fractional Integral Inequalities On Time Scales

Throughout the discussion, $\mathcal{C}(\mathcal{H}, \mathrm{D})$ represents the class of all continuous functions defined on a set $\mathcal{H}$ with range in the set D. Let $\mathbf{R}$ be the set of real numbers, $\mathbb{T}$ be an arbitrary time scale, $\mathfrak{R}$ the set of all regressive and right dense-continuous functions, $\mathfrak{R}^+ = \{p \in \mathfrak{R} : 1 + \mu(\mathfrak{t})p(\mathfrak{t}) > 0,\ \mathfrak{t} \in \mathbb{T}\}$, $[\omega_0, \omega] \subset \mathbf{R}$, $\mathbb{T}_1 := [\omega_0, \omega]_{\mathbb{T}}$, D^a_{Δ,ω_0} the Riemann-Liouville fractional $\Delta-$derivative of order $a > 0$.

Definition 2.3.1 [24] *Let $f : \mathbb{T} \to \mathbf{R}$ is right dense-continuous on $\mathbb{T}$ and $\alpha > 0$, then $\alpha-$Delta integral of f is defined as:*

$$\int_0^{\mathfrak{s}} \mathfrak{f}(\omega)\,(\Delta\omega)^{\alpha} = \Gamma(\alpha+1)\int_0^{\mathfrak{s}} h_{\alpha-1}(\mathfrak{s}, \sigma(\omega))\mathfrak{f}(\omega)\Delta\omega.$$

In particular for $\mathbb{T} = \mathbf{R}$ and $\alpha \in (0, 1]$, the above definition coincides with $[15, Definition\ 4.1]$.

Definition 2.3.2 [24] *Let $q \in \mathbb{N},\ q > 1$ and $\{\tau_1, \tau_2, ..., \tau_q\}$ a set of linearly independent time scales monitored by classical time scales $\mathbb{T}$. Let $f : \mathbb{T}_0^q \to \mathbf{R}^n$ be $rd-$continuous on $\mathbb{T}_0^q$ defined by $f(\mathfrak{s}) = f(\tau_1(\mathfrak{s}), \tau_2(\mathfrak{s}), ..., \tau_q(\mathfrak{s}))$. The $\Delta-$multi-time scale integral of the function f over an interval $[\mathfrak{s}_0, \mathfrak{s}]_{\mathbb{T}} \subseteq \mathbb{T}_0$ is defined as:*

$$(I\mathfrak{f})(\mathfrak{s}) = \int_{\mathfrak{s}_0}^{\mathfrak{s}} \mathfrak{f}(w)\Delta w = \sum_{i=1}^{q}(I_i\mathfrak{f})(\mathfrak{s})$$

provided that:

$$(I_i\mathfrak{f})(\mathfrak{s}) = \int_{\mathfrak{s}_0}^{\mathfrak{s}} \mathfrak{f}(w)\Delta\tau_i(w),\ 1 \le i \le q.$$

In particular for $\mathbb{T} = \mathbf{R}$, the above definition coincides with $[20, Definition\ 3.2]$.

Definition 2.3.3 [24] *Let $W : [0, \mathfrak{s}]_{\mathbb{T}} \times \Psi \to \mathbf{R}$ denote the canonical real valued Wiener process defined up to time $\mathfrak{s} > 0$ and $X : [0, \mathfrak{s}]_{\mathbb{T}} \times \Psi \to \mathbf{R}$ be a stochastic process that is adapted to the natural filtration $\mathcal{G}_*^{\mathfrak{s}}$ of the Wiener process. Then*

$$E\left[\left(\int_0^{\mathfrak{s}} X_t\Delta W_t\right)^2\right] = E\left[\int_0^{\mathfrak{s}} X_t^2\Delta t\right].$$

In particular for $\mathbb{T} = \mathbf{R}$, the above definition coincides with definition of Itô$-$Isometry.

Theorem 2.3.1 [24] *Let $r, g_i : \mathbb{T}_1 \to \mathbf{R}^+$, $1 \le i \le 3$, be nonnegative, right dense-continuous functions which are defined on $\mathbb{T}_1$. Moreover, let $g_j(\mathfrak{t})$, $2 \le j \le 3$, be nondecreasing and bounded by a constant $\mathcal{M} > 0$ such that:*

$$r^{d_1}(\mathfrak{x}) \le g_1(\mathfrak{x}) + g_2(\mathfrak{x})I_{\Delta,\omega_0}r^{d_2}(\mathfrak{x}) + g_3(\mathfrak{x})I^{\alpha}_{\Delta,\omega_0}r^{d_2}(\mathfrak{x}),\ \mathfrak{x} \in \mathbb{T}_1. \tag{2.3.1}$$

Then, for $d_1 \geq d_2 > 0;\ \alpha, \xi > 0;\ \mathfrak{x} \in \mathbb{T}_1;\ \theta, \vartheta \in \mathbb{N}_0,$

$$r(\mathfrak{x}) \leq \sqrt[d_1]{\sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{x}) g_3^{\vartheta}(\mathfrak{x}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta} I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\theta}\tilde{g}_1(\mathfrak{x})}, \tag{2.3.2}$$

provided that:

$$\tilde{g}_1(\mathfrak{x}) := g_1(\mathfrak{x}) + \left(\frac{d_1 - d_2}{d_1}\xi^{\frac{d_2}{d_1}}\right)\{(\mathfrak{x} - \omega_0)g_2(\mathfrak{x}) + h_\alpha(\mathfrak{x}, \omega_0)g_3(\mathfrak{x})\}.$$

Proof. On letting $s_1(\mathfrak{x})$ by

$$s_1(\mathfrak{x}) := g_1(\mathfrak{x}) + g_2(\mathfrak{x}) I_{\Delta,\omega_0} r^{d_2}(\mathfrak{x}) + g_3(\mathfrak{x}) I_{\Delta,\omega_0}^{\alpha} r^{d_2}(\mathfrak{x}), \quad \mathfrak{x} \in \mathbb{T}_1. \tag{2.3.3}$$

Then we have

$$r(\mathfrak{x}) \leq \sqrt[d_1]{s_1(\mathfrak{x})}. \tag{2.3.4}$$

Plugging inequality (2.3.4) in (2.3.3) we have

$$s_1(\mathfrak{x}) \leq g_1(\mathfrak{x}) + g_2(\mathfrak{x}) I_{\Delta,\omega_0} s_1^{\frac{d_2}{d_1}}(\mathfrak{x}) + g_3(\mathfrak{x}) I_{\Delta,\omega_0}^{\alpha} s_1^{\frac{d_2}{d_1}}(\mathfrak{x}) \tag{2.3.5}$$

From (2.3.5), by Lemma 1.5.4 we have

$$\begin{aligned} s_1(\mathfrak{x}) \ &\leq\ g_1(\mathfrak{x}) + g_2(\mathfrak{x}) I_{\Delta,\omega_0}\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} s_1(\mathfrak{x}) + \frac{d_1-d_2}{d_1}\xi^{\frac{d_2}{d_1}}\right) \\ &\quad + g_3(\mathfrak{x}) I_{\Delta,\omega_0}^{\alpha}\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} s_1(\mathfrak{x}) + \frac{d_1-d_2}{d_1}\xi^{\frac{d_2}{d_1}}\right) \\ &=\ \tilde{g}_1(\mathfrak{x}) + \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right) g_2(\mathfrak{x}) I_{\Delta,\omega_0} s_1(\mathfrak{x}) \\ &\quad + \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right) g_3(\mathfrak{x}) I_{\Delta,\omega_0}^{\alpha} s_1(\mathfrak{x}) \end{aligned} \tag{2.3.6}$$

Consider

$$\mathfrak{A}_1\phi(\mathfrak{x}) := \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right) g_2(\mathfrak{x}) I_{\Delta,\omega_0}\phi(\mathfrak{x}) + \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right) g_3(\mathfrak{x}) I_{\Delta,\omega_0}^{\alpha}\phi(\mathfrak{x}), \quad \mathfrak{x} \in \mathbb{T}_1,$$

for right dense-continuous function ϕ, then in this case (2.3.6) is reshaped as:

$$s_1(\mathfrak{x}) \leq \tilde{g}_1(\mathfrak{x}) + \mathfrak{A}_1 s_1(\mathfrak{x})$$

Iterating the inequality for some $\theta \in \mathbb{N}$, one has

$$s_1(\mathfrak{x}) \leq \sum_{\vartheta=0}^{\theta-1}\mathfrak{A}_1^{\vartheta}\tilde{g}_1(\mathfrak{x}) + \mathfrak{A}_1^{\theta} s_1(\mathfrak{x}) \tag{2.3.7}$$

We claim that the following inequality holds:

$$\mathfrak{A}_1^{\theta} s_1(\mathfrak{r}) \leq \sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{r}) g_3^{\vartheta}(\mathfrak{r}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta} I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\theta} s_1(\mathfrak{r}) \quad \text{for some } \theta \in \mathbb{N}. \tag{2.3.8}$$

The proof follows the induction criteria on θ. For $\theta = 1$, the result trivially holds. Suppose it holds for some $\theta = \mathfrak{m}$. Furthermore, if $g_2(\mathfrak{r}), g_3(\mathfrak{r})$ are non-negative and non-decreasing, then, for $\theta = \mathfrak{m} + 1$

$$\begin{aligned}
\mathfrak{A}_1^{\mathfrak{m}+1} s_1(\mathfrak{r}) &= \mathfrak{A}_1(\mathfrak{A}_1^{\mathfrak{m}} s_1(\mathfrak{r})) \\
&\leq \sum_{\vartheta=0}^{\mathfrak{m}} \binom{\mathfrak{m}}{\vartheta} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{r})\right)^{\mathfrak{m}-\vartheta} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathfrak{r})\right)^{\vartheta} \\
&\quad \times \left\{ \frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{r}) I_{\Delta,\omega_0} I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\mathfrak{m}} s_1(\mathfrak{r}) \right. \\
&\quad \left. + \frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathfrak{r}) I_{\Delta,\omega_0}^{\alpha} I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\mathfrak{m}} s_1(\mathfrak{r}) \right\} \\
&= \sum_{\vartheta=0}^{\mathfrak{m}} \binom{\mathfrak{m}}{\vartheta} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{r})\right)^{\mathfrak{m}-\vartheta+1} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathfrak{r})\right)^{\vartheta} \\
&\quad \times I_{\Delta,\omega_0} I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\mathfrak{m}} s_1(\mathfrak{r}) + \sum_{\vartheta=0}^{\mathfrak{m}} \binom{\mathfrak{m}}{\vartheta} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{r})\right)^{\mathfrak{m}-\vartheta} \\
&\quad \times \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathfrak{r})\right)^{\vartheta+1} I_{\Delta,\omega_0}^{\alpha} I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\mathfrak{m}} s_1(\mathfrak{r}) \\
&= \binom{\mathfrak{m}}{0} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{r})\right)^{\mathfrak{m}+1} I_{\Delta,\omega_0}^{\mathfrak{m}+1} s_1(\mathfrak{r}) + \sum_{\vartheta=1}^{\mathfrak{m}} \binom{\mathfrak{m}}{\vartheta} g_2^{\mathfrak{m}-\vartheta+1}(\mathfrak{r}) \\
&\quad \times \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\mathfrak{m}+1} g_3^{\vartheta}(\mathfrak{r}) I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\mathfrak{m}+1} s_1(\mathfrak{r}) + \sum_{\vartheta=1}^{\mathfrak{m}} \binom{\mathfrak{m}}{\vartheta-1} g_2^{\mathfrak{m}-\vartheta+1}(\mathfrak{r}) \\
&\quad \times \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\mathfrak{m}+1} g_3^{\vartheta}(\mathfrak{r}) I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\mathfrak{m}+1} s_1(\mathfrak{r}) \\
&\quad + \binom{\mathfrak{m}}{\mathfrak{m}} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathfrak{r})\right)^{\mathfrak{m}+1} I_{\Delta,\omega_0}^{(\mathfrak{m}+1)\alpha} s_1(\mathfrak{r}) \\
&= \binom{\mathfrak{m}}{0} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{r})\right)^{\mathfrak{m}+1} I_{\Delta,\omega_0}^{\mathfrak{m}+1} s_1(\mathfrak{r}) + \sum_{\vartheta=1}^{\mathfrak{m}} \left[\binom{\mathfrak{m}}{\vartheta-1} + \binom{\mathfrak{m}}{\vartheta}\right] \\
&\quad \times g_2^{\mathfrak{m}-\vartheta+1}(\mathfrak{r}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\mathfrak{m}+1} g_3^{\vartheta}(\mathfrak{r}) I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\mathfrak{m}+1} s_1(\mathfrak{r}) \\
&\quad + \binom{\mathfrak{m}}{\mathfrak{m}} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathfrak{r})\right)^{\mathfrak{m}+1} I_{\Delta,\omega_0}^{(\mathfrak{m}+1)\alpha} s_1(\mathfrak{r})
\end{aligned}$$

which is no more than inequality (2.3.8) for $\theta = \mathfrak{m} + 1$.

We further, claim that $\mathfrak{A}_1^{\theta} s_1(\mathfrak{r}) \to 0$ as $\theta \to \infty$. Consider

$$\mathfrak{I}_{\theta}(\mathfrak{r}) := \sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{r}) g_3^{\vartheta}(\mathfrak{r}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta} I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\theta} s_1(\mathfrak{r}), \ \mathfrak{r} \in \mathbb{T}_1. \tag{2.3.9}$$

Case-I: For $\alpha \in (0,1)$, let $\zeta_\vartheta = \vartheta\alpha - \vartheta + \theta + 1$. Then (ζ_ϑ) is a decreasing sequence on $[0,\theta]$ over $\vartheta \in [0,\theta]$. It may be easily seen that $\max(\zeta_\vartheta) = \theta + 1$; $\min(\zeta_\vartheta) = \theta\alpha + 1$. Furthermore, for a fixed α, there exists a large enough θ_0 such that for any $\theta > \theta_0$, we have $\theta \geq \frac{1}{\alpha}$. So, the sequence satisfies $\zeta_\vartheta \geq 2$ for $\vartheta \in [0,\theta]$. Let $\mathfrak{M}(\mathfrak{x}) = \sup\{s_1(\tau) : \tau \leq \mathfrak{x},\ \mathfrak{x} \in \mathbb{T}_1\}$. Then, without loss of generality and by [4, Theorem 4.2], the equation (2.3.9) can be rewritten as:

$$\begin{aligned}
\mathfrak{I}_\theta(\mathfrak{x}) &\leq \sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{x}) g_3^{\vartheta}(\mathfrak{x}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^\theta \mathfrak{M}(\mathfrak{x}) \frac{(\mathfrak{x}-\omega_0)^{\vartheta\alpha-\vartheta+\theta}}{\Gamma(\vartheta\alpha-\vartheta+\theta+1)} \qquad (2.3.10)\\
&\leq \sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{x}) g_3^{\vartheta}(\mathfrak{x}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^\theta \mathfrak{M}(\mathfrak{x}) \frac{(\mathfrak{x}-\omega_0)^{\vartheta\alpha-\vartheta+\theta}}{\Gamma(\theta\alpha+1)}\\
&= \frac{\mathfrak{M}(\mathfrak{x})}{\Gamma(\theta\alpha+1)} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^\theta \left\{(\mathfrak{x}-\omega_0)g_2(\mathfrak{x}) + (\mathfrak{x}-\omega_0)^\alpha g_3(\mathfrak{x})\right\}^\theta\\
&\leq \frac{\mathfrak{M}(\mathfrak{x})}{\Gamma(\theta\alpha+1)} \left[\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} \left\{(\omega-\omega_0)g_2(\mathfrak{x}) + (\omega-\omega_0)^\alpha g_3(\mathfrak{x})\right\}\right]^\theta.
\end{aligned}$$

Since $g_2(\mathfrak{x}), g_3(\mathfrak{x})$ are bounded and $\Gamma(\theta\alpha+1)$ is growing rapidly for sufficiently large θ, so $\mathfrak{I}_\theta(\mathfrak{x}) \to 0$ for sufficiently large θ and hence, $\mathfrak{A}_1^\theta s_1(\mathfrak{x}) \to 0$. In this case, the inequality (2.3.7) is reshaped as:

$$s_1(\mathfrak{x}) \leq \sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{x}) g_3^{\vartheta}(\mathfrak{x}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^\theta I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\theta}\tilde{g}_1(\mathfrak{x}) \qquad (2.3.11)$$

$$\begin{aligned}
\mathfrak{L}_1(\mathfrak{x};\beta) &:= \sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{x}) g_3^{\vartheta}(\mathfrak{x}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^\theta h_{\vartheta\alpha-\vartheta+\theta}(\beta,\omega_0)\\
&\leq \sum_{\vartheta=0}^{\infty} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathfrak{x})\right)^\vartheta h_{\vartheta\alpha}(\beta,\omega_0) \sum_{\theta=\vartheta}^{\infty} \binom{\theta}{\vartheta} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{x})\right)^{\theta-\vartheta}\\
&\quad \times h_{\theta-\vartheta}(\beta,\omega_0)\\
&\leq \sum_{\vartheta=0}^{\infty} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathfrak{x})\right)^\vartheta h_{\vartheta\alpha}(\beta,\omega_0) \sum_{\theta=\vartheta}^{\infty} \binom{\theta}{\vartheta} \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{x})\right)^{\theta-\vartheta}\\
&\quad \times \frac{(\beta-\omega_0)^{\theta-\vartheta}}{(\theta-\vartheta)!}\\
&= {}_\Delta F_{\alpha,1}\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_3(\mathfrak{x}), \beta, \omega_0\right) \sum_{\mathfrak{p}=0}^{\infty} \frac{1}{\mathfrak{p}!}\binom{\vartheta+\mathfrak{p}}{\vartheta}\\
&\quad \times \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{x})(\beta-\omega_0)\right)^{\mathfrak{p}}\\
&\leq {}_\Delta F_{\alpha,1}\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} \mathcal{M}, \beta, \omega_0\right) \exp\left(\frac{d_2}{\alpha d_1}\xi^{\frac{d_2-d_1}{d_1}} \mathcal{M}(\beta-\omega_0)\right)\\
&=: \mathfrak{L}_1(\mathcal{M};\beta), \qquad (2.3.12)
\end{aligned}$$

provided that

$$\begin{aligned} \binom{\vartheta+\mathfrak{p}}{\vartheta} &= \frac{(\vartheta+\mathfrak{p})!}{\vartheta!\ \mathfrak{p}!} \\ &\leq \frac{(\vartheta+\mathfrak{p})(\vartheta+\mathfrak{p}-1)\cdots(\vartheta+1)}{(\mathfrak{p}-\vartheta\alpha)(\mathfrak{p}-\vartheta\alpha-1)\cdots(1-\vartheta\alpha)} \leq \frac{1}{\alpha^{\mathfrak{p}}}. \end{aligned} \tag{2.3.13}$$

To prove the finiteness of the right hand side of (2.3.2), consider

$$\begin{aligned} \mathfrak{L}_1(\tilde{g_1};\mathfrak{x}) &:= \tilde{g_1}(\mathfrak{x}) + \sum_{\theta=1}^{\infty}\sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{x})g_3^{\vartheta}(\mathfrak{x})\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta} I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\theta}\tilde{g_1}(\mathfrak{x}) \\ &\leq \tilde{g_1}(\mathfrak{x}) + \sum_{\theta=1}^{\infty}\sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta}\mathcal{M}^{\theta}\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta} \\ &\quad \times I_{\Delta,\omega_0}\left(D_{\Delta,\omega_0}\left(h_{\vartheta\alpha-\vartheta+\theta}(\mathfrak{x},\omega_0)\right)\right)\tilde{g_1}(\mathfrak{x}), \end{aligned}$$

hence,

$$\mathfrak{L}_1(\tilde{g_1};\mathfrak{x}) \leq \tilde{g_1}(\mathfrak{x}) + I_{\Delta,\omega_0}\tilde{g_1}(\mathfrak{x})D_{\Delta,\omega_0}\left(\mathfrak{L}_1(\mathcal{M};\mathfrak{x})\right).$$

$\Delta-$Mittag-Leffler function ${}_{\Delta}F_{\alpha,1}$ is an entire function and the exponential function, $\exp(\mathfrak{x})$ is "uniformly continuous" in $\mathfrak{x}$; both $h_{\alpha-1}(\mathfrak{x},\omega_0)$ and $\tilde{g_1}(\mathfrak{x})$ are right dense-continuous for $\mathfrak{x}\in\mathbb{T}_1$. Therefor $\mathfrak{L}_1(\tilde{g_1};\mathfrak{x})<\infty$. A combination of (2.3.4) and (2.3.11) yields the desired result (2.3.2).

Case-II: For $\alpha\geq 1$, let $\eta_{\vartheta}=\vartheta\alpha-\vartheta+\theta+1$. Then (η_{ϑ}) is non-decreasing sequence on $[0,\theta]$ over $\vartheta\in[0,\theta]$. It may be easily seen that $\max(\eta_{\vartheta})=\theta\alpha+1;\ \min(\eta_{\vartheta})=\theta+1$ and $\eta_{\vartheta}\in[2,\infty)$. Moreover, from inequality (2.3.10) we have

$$\begin{aligned} \mathfrak{I}_{\theta}(\mathfrak{x}) &\leq \sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{x})g_3^{\vartheta}(\mathfrak{x})\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta}\mathfrak{M}(\mathfrak{x})\frac{(\mathfrak{x}-\omega_0)^{\vartheta\alpha-\vartheta+\theta}}{\Gamma(\vartheta\alpha-\vartheta+\theta+1)} \\ &\leq \sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{x})g_3^{\vartheta}(\mathfrak{x})\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta}\mathfrak{M}(\mathfrak{x})\frac{(\mathfrak{x}-\omega_0)^{\vartheta\alpha-\vartheta+\theta}}{\Gamma(\theta+1)} \\ &\leq \frac{\mathfrak{M}(\mathfrak{x})}{\Gamma(\theta+1)}\left[\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\left\{(\omega-\omega_0)g_2(\mathfrak{x})+(\omega-\omega_0)^{\alpha}g_3(\mathfrak{x})\right\}\right]^{\theta}. \end{aligned}$$

Since $g_2(\mathfrak{x}), g_3(\mathfrak{x})$ are bounded and $\Gamma(\theta+1)$ is growing rapidly for sufficiently large θ, so $\mathfrak{I}_{\theta}(\mathfrak{x})\to 0$ for sufficiently large θ and hence, $\mathfrak{A}_1^{\theta}s_1(\mathfrak{x})\to 0$. Again, in this case the inequality (2.3.7) reduces to inequality (2.3.11). Further,

$$\begin{aligned} \mathfrak{L}_2(\mathfrak{x};\beta) &:= \sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{x})g_3^{\vartheta}(\mathfrak{x})\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta} h_{\vartheta\alpha-\vartheta+\theta}(\beta,\omega_0) \\ &\leq {}_{\Delta}F_{\alpha,1}\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_3(\mathfrak{x}),\beta,\omega_0\right)\sum_{\mathfrak{p}=0}^{\infty}\frac{1}{\mathfrak{p}!}\binom{\vartheta+\mathfrak{p}}{\vartheta} \end{aligned}$$

$$\begin{aligned} &\times \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_2(\mathfrak{x})(\beta-\omega_0)\right)^{\mathfrak{p}} \\ \leq\ & {}_{\Delta}F_{\alpha,1}\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\mathcal{M},\beta,\omega_0\right)\exp\left(\frac{\alpha d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\mathcal{M}(\beta-\omega_0)\right) \\ =:\ & \mathfrak{L}_2(\mathcal{M};\beta), \end{aligned}$$

provided that

$$\begin{aligned} \binom{\vartheta+\mathfrak{p}}{\vartheta} &= \frac{(\vartheta+\mathfrak{p})!}{\vartheta!\ \mathfrak{p}!} \\ &\leq \frac{(\vartheta+\mathfrak{p})(\vartheta+\mathfrak{p}-1)\cdots(\vartheta+1)}{(\mathfrak{p}-\frac{\vartheta}{\alpha})(\mathfrak{p}-\frac{\vartheta}{\alpha}-1)\cdots(1-\frac{\vartheta}{\alpha})} \leq \alpha^{\mathfrak{p}}. \end{aligned}$$

Repeating the same steps as in Case-I, the finiteness of the right hand side of (2.3.2) can be proved. □

Remark 2.3.1 *For* $\mathbb{T}=\mathbf{R}$; $d_1=1=d_2$; $\alpha\in(0,1)$; $\omega_0=0$; $g_3(\mathfrak{x})=g(\mathfrak{x})\Gamma(\alpha)$, *Theorem* 2.3.1 *coincides with [27, Theorem 2.1].*

The following result is the discretization of the Theorem 2.3.1.

Corollary 2.3.2 [24] *Let* $g_i,\ 1\leq i\leq 3,$ *and* r *be non-negative real valued functions defined on* $\mathbb{N}_0$. *Furthermore, if* $g_j,\ 2\leq j\leq 3,$ *is nondecreasing and bounded such that*

$$r^{d_1}(\mathfrak{x})\leq g_1(\mathfrak{x})+g_2(\mathfrak{x})\ \Delta_0^{-1}\ r^{d_2}(\mathfrak{x})+g_3(\mathfrak{x})\ \Delta_0^{-n}\ r^{d_2}(\mathfrak{x}),\qquad \mathfrak{x}\in\mathbb{N}_0.$$

Then, for $d_1\geq d_2>0,\ \xi>0,\ n\in\mathbb{N},\ \mathfrak{x},\theta,\vartheta\in\mathbb{N}_0,$ *we have*

$$r(\mathfrak{x})\leq\sqrt[d_1]{\sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta}g_2^{\theta-\vartheta}(\mathfrak{x})g_3^{\vartheta}(\mathfrak{x})\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}\right)^{\theta}\ \Delta_0^{-\vartheta n+\vartheta-\theta}\tilde{g}_2(\mathfrak{x})},$$

provided that

$$\tilde{g}_2(\mathfrak{x}):=g_1(\mathfrak{x})+\left(\frac{d_1-d_2}{d_1}\xi^{\frac{d_2}{d_1}}\right)\left\{\mathfrak{x}g_2(\mathfrak{x})+\frac{\mathfrak{x}^n}{\Gamma(n+1)}g_3(\mathfrak{x})\right\}.$$

Theorem 2.3.3 [24] *Let the conditions of Theorem* 2.3.1 *be satisfied, for* $d_1\geq 1$. *Let* $L:\mathbb{T}_0\times\mathbf{R}^+\to\mathbf{R}^+$ *be nonnegative, right dense-continuous on* $\mathbb{T}_0$ *and continuous on* $\mathbf{R}^+$, *with* $0\leq \mathrm{L}(\mathfrak{x},r)-\mathrm{L}(\mathfrak{x},s)\leq\mathcal{K}(r-s)$ *for* $r\geq s\geq 0$, *where* $\mathcal{K}$ *is the Lipschitz constant such that*

$$r^{d_1}(\mathfrak{x})\leq g_1(\mathfrak{x})+g_2(\mathfrak{x})I_{\Delta,\omega_0}\mathrm{L}\left(\mathfrak{x},r(\mathfrak{x})\right)+g_3(\mathfrak{x})I^{\alpha}_{\Delta,\omega_0}\mathrm{L}\left(\mathfrak{x},r(\mathfrak{x})\right),\ \mathfrak{x}\in\mathbb{T}_1. \tag{2.3.14}$$

Then

$$r(\mathfrak{x}) \leq \sqrt[d_1]{\sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{x}) g_3^{\vartheta}(\mathfrak{x}) \left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)^{\theta} I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\theta}\tilde{g}_3(\mathfrak{x})}, \tag{2.3.15}$$

provided that

$$\begin{aligned}\tilde{g}_3(\mathfrak{x}) &:= g_1(\mathfrak{x}) + g_2(\mathfrak{x}) I_{\Delta,\omega_0}\mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\\ &\quad + g_3(\mathfrak{x}) I_{\Delta,\omega_0}^{\alpha}\mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right).\end{aligned}$$

Proof. On letting the right hand side of (2.3.14) by $s_2(\mathfrak{x})$, we have

$$r(\mathfrak{x}) \leq \sqrt[d_1]{s_2(\mathfrak{x})}. \tag{2.3.16}$$

Further,

$$s_2(\mathfrak{x}) \leq g_1(\mathfrak{x}) + g_2(\mathfrak{x}) I_{\Delta,\omega_0}\mathrm{L}\left(\mathfrak{x}, \sqrt[d_1]{s_2(\mathfrak{x})}\right) + g_3(\mathfrak{x}) I_{\Delta,\omega_0}^{\alpha}\mathrm{L}\left(\mathfrak{x}, \sqrt[d_1]{s_2(\mathfrak{x})}\right) \tag{2.3.17}$$

From (2.3.17), by Lemma 1.5.4 we have

$$\begin{aligned} s_2(\mathfrak{x}) &\leq g_1(\mathfrak{x}) + g_2(\mathfrak{x}) I_{\Delta,\omega_0}\mathrm{L}\left(\mathfrak{x}, \frac{1}{d_1}\xi^{\frac{1-d_1}{d_1}} s_2(\mathfrak{x}) + \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\\ &\quad + g_3(\mathfrak{x}) I_{\Delta,\omega_0}^{\alpha}\mathrm{L}\left(\mathfrak{x}, \frac{1}{d_1}\xi^{\frac{1-d_1}{d_1}} s_2(\mathfrak{x}) + \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\\ &= g_1(\mathfrak{x}) + g_2(\mathfrak{x}) I_{\Delta,\omega_0}\left\{\mathrm{L}\left(\mathfrak{x}, \frac{1}{d_1}\xi^{\frac{1-d_1}{d_1}} s_2(\mathfrak{x}) + \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\right.\\ &\quad \left. -\mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right) + \mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\right\}\\ &\quad + g_3(\mathfrak{x}) I_{\Delta,\omega_0}^{\alpha}\left\{\mathrm{L}\left(\mathfrak{x}, \frac{1}{d_1}\xi^{\frac{1-d_1}{d_1}} s_2(\mathfrak{x}) + \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\right.\\ &\quad \left. -\mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right) + \mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\right\} \end{aligned} \tag{2.3.18}$$

From (2.3.18), by Lipschitz continuity on L we get

$$\begin{aligned} s_2(\mathfrak{x}) &\leq g_1(\mathfrak{x}) + g_2(\mathfrak{x}) I_{\Delta,\omega_0}\left\{\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}} s_2(\mathfrak{x}) + \mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\right\}\\ &\quad + g_3(\mathfrak{x}) I_{\Delta,\omega_0}^{\alpha}\left\{\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}} s_2(\mathfrak{x}) + \mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right)\right\} \end{aligned}$$

$$\begin{aligned} s_2(\mathfrak{x}) &\leq \tilde{g}_3(\mathfrak{x}) + \left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right) g_2(\mathfrak{x}) I_{\Delta,\omega_0} s_2(\mathfrak{x})\\ &\quad + \left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right) g_3(\mathfrak{x}) I_{\Delta,\omega_0}^{\alpha} s_2(\mathfrak{x}). \end{aligned} \tag{2.3.19}$$

Consider

$$\mathfrak{A}_2\phi(\mathfrak{x}) := \left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right) g_2(\mathfrak{x}) I_{\Delta,\omega_0}\phi(\mathfrak{x}) + \left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right) g_3(\mathfrak{x}) I^{\alpha}_{\Delta,\omega_0}\phi(\mathfrak{x}),$$

for right dense-continuous function $\phi(\mathfrak{x})$ such that $\mathfrak{x} \in \mathbb{T}_1$. Then, in this case (2.3.19) is reshaped as:

$$s_2(\mathfrak{x}) \leq \tilde{g}_3(\mathfrak{x}) + \mathfrak{A}_2 s_2(\mathfrak{x})$$

Iterating the inequality for some $\theta \in \mathbb{N}$, one has

$$s_2(\mathfrak{x}) \leq \sum_{\vartheta=0}^{\theta-1} \mathfrak{A}_2^{\vartheta}\tilde{g}_3(\mathfrak{x}) + \mathfrak{A}_2^{\theta} s_2(\mathfrak{x})$$

We claim that the following inequality holds:

$$\mathfrak{A}_2^{\theta} s_2(\mathfrak{x}) \leq \sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{x}) g_3^{\vartheta}(\mathfrak{x}) \left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)^{\theta} I^{\vartheta\alpha-\vartheta+\theta}_{\Delta,\omega_0} s_2(\mathfrak{x})$$

for some $\theta \in \mathbb{N}$. This can be proved by following the parallel steps beyond the inequality (2.3.8). Ultimately, we get the inequality (2.3.15). $\square$

Remark 2.3.2 *For* $\mathbb{T} = \mathbf{R}$; $\omega_0 = 0$; $g_2(\mathfrak{x}) \equiv 0$, *Theorem* 2.3.3 *coincides with [10, Theorem 2].*

Corollary 2.3.4 [24] *Let* g_i, $1 \leq i \leq 3$, *and* r *be non-negative real valued functions defined on* $\mathbb{N}_0$. *Let* L *be non-negative real valued function defined on* $\mathbb{N}_0 \times \mathbf{R}^+$ *such that* $0 \leq \mathrm{L}(\mathfrak{x}, r) - \mathrm{L}(\mathfrak{x}, s) \leq \mathcal{K}(r-s)$ *for* $r \geq s \geq 0$ *and* $\mathcal{K} > 0$. *Moreover, if* $g_2(\mathfrak{x})$ *and* $g_3(\mathfrak{x})$ *are nondecreasing and bounded such that*

$$r^{d_1}(\mathfrak{x}) \leq g_1(\mathfrak{x}) + g_2(\mathfrak{x})\Delta_0^{-1}\mathrm{L}\left(\mathfrak{x}, r(\mathfrak{x})\right) + g_3(\mathfrak{x})\Delta_0^{-n}\mathrm{L}\left(\mathfrak{x}, r(\mathfrak{x})\right),\ \mathfrak{x} \in \mathbb{N}_0,$$

then, for $d_1 \geq 1$, $\xi > 0$, $n \in \mathbb{N}$, $t, \theta, \vartheta \in \mathbb{N}_0$,

$$r(\mathfrak{x}) \leq \sqrt[d_1]{\sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} g_2^{\theta-\vartheta}(\mathfrak{x}) g_3^{\vartheta}(\mathfrak{x}) \left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)^{\theta} \Delta_0^{-\vartheta n+\vartheta-\theta}\tilde{g}_4(\mathfrak{x})},$$

provided that

$$\begin{aligned}\tilde{g}_4(\mathfrak{x}) \;&:=\; g_1(\mathfrak{x}) + g_2(\mathfrak{x})\Delta_0^{-1}\mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right) \\ &\quad + g_3(\mathfrak{x})\Delta_0^{-n}\mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right).\end{aligned}$$

Theorem 2.3.5 [24] *Let the conditions of Theorem* 2.3.3 *be satisfied. Moreover, if* g_1 *is non-decreasing;* $g_4 : \mathbb{T}_1 \to \mathbf{R}^+$ *is nonnegative and right dense-continuous on* $\mathbb{T}_1$*, with* $d_1 \geq d_2 \geq 1$ *such that*

$$\begin{aligned} r^{d_1}(\mathfrak{x}) &\leq g_1(\mathfrak{x}) + g_2(\mathfrak{x}) I_{\Delta,\omega_0} \mathrm{L}\left(\mathfrak{x}, r(\mathfrak{x})\right) + g_3(\mathfrak{x}) I^{\alpha}_{\Delta,\omega_0} \mathrm{L}\left(\mathfrak{x}, r(\mathfrak{x})\right) \\ &\quad + I_{\Delta,\omega_0} g_4(\mathfrak{x}) r^{d_2}(\mathfrak{x}), \;\; \mathfrak{x} \in \mathbb{T}_1 \end{aligned} \tag{2.3.20}$$

then

$$\begin{aligned} r(\mathfrak{x}) \leq & \sqrt[d_1]{e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_4}(\mathfrak{x}, \omega_0)} \times \\ & \sqrt[d_1]{\sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta} \binom{\theta}{\vartheta} \tilde{g}_{4,2}^{\theta-\vartheta}(\mathfrak{x}) \tilde{g}_{4,3}^{\vartheta}(\mathfrak{x}) \left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)^{\theta} I^{\vartheta\alpha-\vartheta+\theta}_{\Delta,\omega_0} \tilde{g}_5(\mathfrak{x})} \end{aligned} \tag{2.3.21}$$

provided that

$$\begin{aligned} \tilde{g}_5(\mathfrak{x}) &:= g_1(\mathfrak{x}) + g_2(\mathfrak{x}) I_{\Delta,\omega_0} \mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right) \\ &\quad + g_3(\mathfrak{x}) I^{\alpha}_{\Delta,\omega_0} \mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right) \\ &\quad + \frac{d_1-d_2}{d_1}\xi^{\frac{d_2}{d_1}} I_{\Delta,\omega_0} g_4(\mathfrak{x}). \\ \tilde{g}_{4,j}(\mathfrak{x}) &:= e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_4}(\mathfrak{x}, \omega_0)\; g_j(\mathfrak{x}), \quad 2 \leq j \leq 3. \end{aligned}$$

Proof. On letting the right hand side of (2.3.20) by $s_3(\mathfrak{x})$, we have

$$r(\mathfrak{x}) \leq \sqrt[d_1]{s_3(\mathfrak{x})}. \tag{2.3.22}$$

Further,

$$\begin{aligned} s_3(\mathfrak{x}) &\leq g_1(\mathfrak{x}) + g_2(\mathfrak{x}) I_{\Delta,\omega_0} \mathrm{L}\left(\mathfrak{x}, \sqrt[d_1]{s_3(\mathfrak{x})}\right) + g_3(\mathfrak{x}) I^{\alpha}_{\Delta,\omega_0} \mathrm{L}\left(\mathfrak{x}, \sqrt[d_1]{s_3(\mathfrak{x})}\right) \\ &\quad + I_{\Delta,\omega_0} g_4(\mathfrak{x}) \left(s_3(\mathfrak{x})\right)^{\frac{d_2}{d_1}} \end{aligned} \tag{2.3.23}$$

From (2.3.23), by Lemma 1.5.4 we have

$$\begin{aligned} s_3(\mathfrak{x}) &\leq g_1(\mathfrak{x}) + g_2(\mathfrak{x}) I_{\Delta,\omega_0} \mathrm{L}\left(\mathfrak{x}, \frac{1}{d_1}\xi^{\frac{1-d_1}{d_1}} s_3(\mathfrak{x}) + \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right) \\ &\quad + g_3(\mathfrak{x}) I^{\alpha}_{\Delta,\omega_0} \mathrm{L}\left(\mathfrak{x}, \frac{1}{d_1}\xi^{\frac{1-d_1}{d_1}} s_3(\mathfrak{x}) + \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right) \\ &\quad + I_{\Delta,\omega_0}\left\{ g_4(\mathfrak{x}) \left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} s_3(\mathfrak{x}) + \frac{d_1-d_2}{d_1}\xi^{\frac{d_2}{d_1}}\right)\right\} \end{aligned} \tag{2.3.24}$$

From (2.3.24), by Lipschitz continuity on L we obtain

$$\begin{aligned} s_3(\mathfrak{x}) &\leq g_1(\mathfrak{x}) + g_2(\mathfrak{x}) I_{\Delta,\omega_0} \left\{ \frac{\mathcal{K}}{d_1} \xi^{\frac{1-d_1}{d_1}} s_3(\mathfrak{x}) + \mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1} \sqrt[d_1]{\xi}\right) \right\} \\ &\quad + g_3(\mathfrak{x}) I^{\alpha}_{\Delta,\omega_0} \left\{ \frac{\mathcal{K}}{d_1} \xi^{\frac{1-d_1}{d_1}} s_3(\mathfrak{x}) + \mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1} \sqrt[d_1]{\xi}\right) \right\} \\ &\quad + \frac{d_2}{d_1} \xi^{\frac{d_2-d_1}{d_1}} I_{\Delta,\omega_0} g_4(\mathfrak{x}) s_3(\mathfrak{x}) + \frac{d_1-d_2}{d_1} \xi^{\frac{d_2}{d_1}} I_{\Delta,\omega_0} g_4(\mathfrak{x}) \\ &= \mathfrak{Z}(\mathfrak{x}) + \frac{d_2}{d_1} \xi^{\frac{d_2-d_1}{d_1}} I_{\Delta,\omega_0} g_4(\mathfrak{x}) s_3(\mathfrak{x}), \quad \mathfrak{x} \in \mathbb{T}_0, \end{aligned} \tag{2.3.25}$$

provided that

$$\begin{aligned} \mathfrak{Z}(\mathfrak{x}) &:= g_1(\mathfrak{x}) + g_2(\mathfrak{x}) I_{\Delta,\omega_0} \left\{ \frac{\mathcal{K}}{d_1} \xi^{\frac{1-d_1}{d_1}} s_3(\mathfrak{x}) + \mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1} \sqrt[d_1]{\xi}\right) \right\} \\ &\quad + g_3(\mathfrak{x}) I^{\alpha}_{\Delta,\omega_0} \left\{ \frac{\mathcal{K}}{d_1} \xi^{\frac{1-d_1}{d_1}} s_3(\mathfrak{x}) + \mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1} \sqrt[d_1]{\xi}\right) \right\} \\ &\quad + \frac{d_1-d_2}{d_1} \xi^{\frac{d_2}{d_1}} I_{\Delta,\omega_0} g_4(\mathfrak{x}). \end{aligned}$$

From (2.3.25), by Theorem 1.2.7 we have

$$\begin{aligned} s_3(\mathfrak{x}) &\leq \mathfrak{Z}(\mathfrak{x}) + \int_{\omega_0}^{\mathfrak{x}} e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_4}(\mathfrak{x}, \sigma(w)) \mathfrak{Z}(w) \frac{d_2}{d_1} \xi^{\frac{d_2-d_1}{d_1}} g_4(w) \Delta w \\ &\leq \mathfrak{Z}(\mathfrak{x}) + \mathfrak{Z}(\mathfrak{x}) \int_{\omega_0}^{\mathfrak{x}} e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_4}(\mathfrak{x}, \sigma(w)) \frac{d_2}{d_1} \xi^{\frac{d_2-d_1}{d_1}} g_4(w) \Delta w. \end{aligned} \tag{2.3.26}$$

From (2.3.26) by Theorems 1.2.8 we have

$$\begin{aligned} s_3(\mathfrak{x}) &\leq \mathfrak{Z}(\mathfrak{x}) + \mathfrak{Z}(\mathfrak{x}) \left\{ e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_4}(\mathfrak{x}, \omega_0) - e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_4}(\mathfrak{x}, \mathfrak{x}) \right\} \\ &= \mathfrak{Z}(\mathfrak{x}) + \mathfrak{Z}(\mathfrak{x}) \left(e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_4}(\mathfrak{x}, \omega_0) - 1 \right) \\ &= \mathfrak{Z}(\mathfrak{x}) e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_4}(\mathfrak{x}, \omega_0), \end{aligned}$$

and hence,

$$\begin{aligned} \mathfrak{Z}(\mathfrak{x}) &\leq g_1(\mathfrak{x}) + g_2(\mathfrak{x}) I_{\Delta,\omega_0} \left\{ \frac{\mathcal{K}}{d_1} \xi^{\frac{1-d_1}{d_1}} \mathfrak{Z}(\mathfrak{x}) e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_4}(\mathfrak{x}, \omega_0) \right. \\ &\quad \left. + \mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1} \sqrt[d_1]{\xi}\right) \right\} + g_3(\mathfrak{x}) I^{\alpha}_{\Delta,\omega_0} \left\{ \frac{\mathcal{K}}{d_1} \xi^{\frac{1-d_1}{d_1}} \mathfrak{Z}(\mathfrak{x}) \right. \\ &\quad \left. \times e_{\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}} g_4}(\mathfrak{x}, \omega_0) + \mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1} \sqrt[d_1]{\xi}\right) \right\} \\ &\quad + \frac{d_1-d_2}{d_1} \xi^{\frac{d_2}{d_1}} I_{\Delta,\omega_0} g_4(\mathfrak{x}) \end{aligned}$$

$$\leq \tilde{g}_5(\mathfrak{x}) + \frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\tilde{g}_{4,2}(\mathfrak{x})I_{\Delta,\omega_0}\mathfrak{Z}(\mathfrak{x}) + \frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\tilde{g}_{4,3}(\mathfrak{x}) \times I^{\alpha}_{\Delta,\omega_0}\mathfrak{Z}(\mathfrak{x}). \quad (2.3.27)$$

Consider

$$\mathfrak{A}_3\phi(\mathfrak{x}) := \left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)\tilde{g}_{4,2}(\mathfrak{x})I_{\Delta,\omega_0}\phi(\mathfrak{x}) + \left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)\tilde{g}_{4,3}(\mathfrak{x})I^{\alpha}_{\Delta,\omega_0}\phi(\mathfrak{x}),$$

for right dense-continuous function $\phi(\mathfrak{x})$ such that $\mathfrak{x} \in \mathbb{T}_1$. Then, in this case (2.3.27) is reshaped as:

$$\mathfrak{Z}(\mathfrak{x}) \leq \tilde{g}_5(\mathfrak{x}) + \mathfrak{A}_3\mathfrak{Z}(\mathfrak{x})$$

Iterating the inequality for some $\theta \in \mathbb{N}$, one has

$$\mathfrak{Z}(\mathfrak{x}) \leq \sum_{\vartheta=0}^{\theta-1}\mathfrak{A}_3^{\vartheta}\tilde{g}_5(\mathfrak{x}) + \mathfrak{A}_3^{\theta}\mathfrak{Z}(\mathfrak{x})$$

We claim that the following inequality holds:

$$\mathfrak{A}_3^{\theta}\mathfrak{Z}(\mathfrak{x}) \leq \sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta}\tilde{g}_{4,2}^{\theta-\vartheta}(\mathfrak{x})\tilde{g}_{4,3}^{\vartheta}(\mathfrak{x})\left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)^{\theta} I_{\Delta,\omega_0}^{\vartheta\alpha-\vartheta+\theta}\mathfrak{Z}(\mathfrak{x})$$

for some $\theta \in \mathbb{N}$. This can be proved by following the parallel steps beyond the inequality (2.3.8). Ultimately, we get the inequality (2.3.21). □

Remark 2.3.3 *For* $\mathbb{T} = \mathbf{R}$; $\omega_0 = 0$; $g_2(\mathfrak{x}) \equiv 0$, *Theorem* 2.3.5 *coincides with [10, Theorem 4].*

Corollary 2.3.6 [24] *Let* $g_k,\ 1 \leq k \leq 4$, *and* r *be non-negative real valued function defined on* $\mathbb{N}_0$. *Let* L *be non-negative real valued function defined on* $\mathbb{N}_0 \times \mathbf{R}^+$ *such that* $0 \leq \mathrm{L}(\mathfrak{x}, r) - \mathrm{L}(\mathfrak{x}, s) \leq \mathcal{K}(r - s)$ *for* $r \geq s \geq 0$ *and* $\mathcal{K} > 0$. *Moreover, if* $g_i,\ 1 \leq i \leq 3$, *is nondecreasing such that*

$$r^{d_1}(\mathfrak{x}) \leq g_1(\mathfrak{x}) + g_2(\mathfrak{x})\Delta_0^{-1}\mathrm{L}(\mathfrak{x}, r(\mathfrak{x})) + g_3(\mathfrak{x})\Delta_0^{-n}\mathrm{L}(\mathfrak{x}, r(\mathfrak{x})) + \Delta_0^{-1}g_4(\mathfrak{x})r^{d_2}(\mathfrak{x}),\ \mathfrak{x} \in \mathbb{N}_0.$$

Then, for $d_1 \geq d_2 \geq 1,\ \xi > 0,\ n \in \mathbb{N},\ \mathfrak{x}, x, \theta, \vartheta \in \mathbb{N}_0$,

$$r(\mathfrak{x}) \leq \sqrt[d_1]{\prod_{x=0}^{\mathfrak{x}-1}\left\{1 + \frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4(x)\right\}} \times \sqrt[d_1]{\sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta}\tilde{\tilde{g}}_{4,2}^{\theta-\vartheta}(\mathfrak{x})\tilde{\tilde{g}}_{4,3}^{\vartheta}(\mathfrak{x})\left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)^{\theta}\Delta_0^{-\vartheta n+\vartheta-\theta}\tilde{g}_6(\mathfrak{x})},$$

provided that

$$\begin{aligned}
\tilde{g}_6(\mathfrak{x}) &:= g_1(\mathfrak{x}) + g_2(\mathfrak{x})\Delta_0^{-1}\mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right) \\
&\quad + g_3(\mathfrak{x})\Delta_0^{-n}\mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right) \\
&\quad + \frac{d_1-d_2}{d_1}\xi^{\frac{d_2}{d_1}}\Delta_0^{-1}g_4(\mathfrak{x}). \\
\tilde{\tilde{g}}_{4,j}(\mathfrak{x}) &:= \prod_{x=0}^{\mathfrak{x}-1}\left\{1 + \frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4(x)\right\} g_j(\mathfrak{x}),\ 2 \le j \le 3.
\end{aligned}$$

Theorem 2.3.7 [24] *Let the conditions of Theorem* 2.3.3 *be satisfied. Moreover, if* g_1 *is non-decreasing;* $g_4, g_5 : \mathbb{T}_1 \to \mathbf{R}^+$ *are nonnegative and right dense-continuous on* $\mathbb{T}_1$, *with* $d_1 \ge d_2 \ge 1$, $d_1 \ge d_3 \ge 1$, *such that*

$$\begin{aligned}
r^{d_1}(\mathfrak{x}) &\le g_1(\mathfrak{x}) + g_2(\mathfrak{x})I_{\Delta,\omega_0}\mathrm{L}(\mathfrak{x}, r(\mathfrak{x})) + g_3(\mathfrak{x})I^{\alpha}_{\Delta,\omega_0}\mathrm{L}(\mathfrak{x}, r(\mathfrak{x})) \\
&\quad + I_{\Delta,\omega_0}g_4(\mathfrak{x})r^{d_2}(\mathfrak{x}) + I_{\Delta,\omega_0}g_5(\mathfrak{x})r^{d_3}(\mathfrak{x}), \quad \mathfrak{x} \in \mathbb{T}_1.
\end{aligned}$$

Then,

$$\begin{aligned}
r(\mathfrak{x}) &\le \sqrt[d_1]{e_{\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4 + \frac{d_3}{d_1}\xi^{\frac{d_3-d_1}{d_1}}g_5\right)}(\mathfrak{x}, \omega_0)} \\
&\quad \times \sqrt[d_1]{\sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta}\tilde{g}_{5,2}^{\theta-\vartheta}(\mathfrak{x})\tilde{g}_{5,3}^{\vartheta}(\mathfrak{x})\left(\frac{\mathcal{K}}{d_1}\xi^{\frac{1-d_1}{d_1}}\right)^{\theta} I^{\vartheta\alpha-\vartheta+\theta}_{\Delta,\omega_0}\tilde{g}_7(\mathfrak{x})}
\end{aligned}$$

for $\mathfrak{x} \in \mathbb{T}_1$, *provided that*

$$\begin{aligned}
\tilde{g}_7(\mathfrak{x}) &:= g_1(\mathfrak{x}) + g_2(\mathfrak{x})I_{\Delta,\omega_0}\mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right) \\
&\quad + g_3(\mathfrak{x})I^{\alpha}_{\Delta,\omega_0}\mathrm{L}\left(\mathfrak{x}, \frac{d_1-1}{d_1}\sqrt[d_1]{\xi}\right) \\
&\quad + \frac{d_1-d_2}{d_1}\xi^{\frac{d_2}{d_1}}I_{\Delta,\omega_0}g_4(\mathfrak{x}) + \frac{d_1-d_3}{d_1}\xi^{\frac{d_3}{d_1}}I_{\Delta,\omega_0}g_5(\mathfrak{x}). \\
\tilde{g}_{5,j}(\mathfrak{x}) &:= e_{\left(\frac{d_2}{d_1}\xi^{\frac{d_2-d_1}{d_1}}g_4 + \frac{d_3}{d_1}\xi^{\frac{d_3-d_1}{d_1}}g_5\right)}(\mathfrak{x}, \omega_0)\ g_j(\mathfrak{x}),\ ,\ 2 \le j \le 3.
\end{aligned}$$

CHAPTER 3

Analysis of solutions of certain type of differential equations

This chapter shows that fractional and dynamical integral inequalities can be used as powerful tools in the qualitative analysis of the solutions of some certain fractional and dynamic problems. We also discussed that integral inequalities are helpful to find the solutions of complicated phenomena in a more descriptive and compact form. Section 3.1 shows the existence and uniqueness of the solution of fractional stochastic differential equation. Section 3.2, check the behavior of the solutions of some certain dynamic equations with initial conditions. Moreover, numerical example has been discussed to ensure the validity of the derived results. Section 3.3 shows the boundedness and uniqueness of Cauchy type problem. In Section 3.4, we structured the fractional $\Delta-$stochastic differential equation of It$\hat{o}-$Doob type and check the behaviour of the solutions of nonlinear fractional $\Delta-$stochastic differential equation.

3.1 Fractional stochastic differential equation

Consider the following stochastic differential equation:

$$d(A(\mathfrak{x})) = b(\mathfrak{x}, A(\mathfrak{x}))d\mathfrak{x} + \sigma_1(\mathfrak{x}, A(\mathfrak{x}))d\mathfrak{x}^a + \sigma_2(\mathfrak{x}, A(\mathfrak{x}))dB_{\mathfrak{x}} \tag{3.1.1}$$

where $0 < a < 1$ and $B_{\mathfrak{x}}$ is the standard Brownian motion.

Theorem 3.1.1 *[22] Let $\omega > 0$; $a \in (0,1)$; $(\Omega, \mathcal{G}, \rho)$ be a complete probability space with an $m-$dimensional Brownian motion $B(\mathfrak{x})$ defined on space $\mathbb{R}^n$; let w_0 be a random variable such that $E|w_0|^2 < \infty$; let $b(.,.), \sigma_1(.,.) : [0,\omega] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma_2(.,.) : [0,\omega] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions such that $\mathfrak{x}^{1-a}b(.,.)$, $\mathfrak{x}^{1-a}\sigma_1(.,.)$, $\mathfrak{x}^{1-a}\sigma_2(.,.)$ are also measurable such that the linear Growth and Lipschitz conditions,*

$$|b(\mathfrak{x}, A)|^2 + |\sigma_1(\mathfrak{x}, A)|^2 + |\sigma_2(\mathfrak{x}, A)|^2 \leq K^2\left(1 + |A|^2\right) \tag{3.1.2}$$

$$|b(\mathfrak{x}, A) - b(\mathfrak{x}, \mathfrak{y})| + |\sigma_1(\mathfrak{x}, A) - \sigma_1(\mathfrak{x}, \mathfrak{y})| + |\sigma_2(\mathfrak{x}, A) - \sigma_2(\mathfrak{x}, \mathfrak{y})| \leq L|A - \mathfrak{y}| \tag{3.1.3}$$

are satisfied, for some constants $K, L > 0$. Then the fractional stochastic differential equation (3.1.1) has a $\mathfrak{x}-$continuous solution with a filtration $\mathcal{G}_{\mathfrak{x}}^{w_0}$ such that

$$E\left[\int_0^\omega |A(\mathfrak{x})|^2 d\mathfrak{x}\right] < \infty.$$

Proof. The integral form of the stochastic differential equation (3.1.1) is

$$\begin{aligned} A(\mathfrak{x}) &= w_0 + \int_0^{\mathfrak{x}} b\left(\mathfrak{s}, A(\mathfrak{s})\right) d\mathfrak{s} + a\int_0^{\mathfrak{x}} (\mathfrak{x}-\mathfrak{s})^{a-1}\sigma_1\left(\mathfrak{s}, A(\mathfrak{s})\right) d\mathfrak{s} \\ &\quad + \int_0^{\mathfrak{x}} \sigma_2\left(\mathfrak{s}, A(\mathfrak{s})\right) dB_{\mathfrak{s}} \end{aligned} \tag{3.1.4}$$

By the method of Picard-Lindelöf iteration, define logarithmically $A^{(0)}(\mathfrak{x}) = A_0$, for some $\eta \in N$, as follows:

$$\begin{aligned} A^{(\eta+1)}(\mathfrak{x}) &= w_0 + \int_0^{\mathfrak{x}} b\left(\mathfrak{s}, A^{(\eta)}(\mathfrak{s})\right) d\mathfrak{s} + a\int_0^{\mathfrak{x}} (\mathfrak{x}-\mathfrak{s})^{a-1}\sigma_1\left(\mathfrak{s}, A^{(\eta)}(\mathfrak{s})\right) d\mathfrak{s} \\ &\quad + \int_0^{\mathfrak{x}} \sigma_2\left(\mathfrak{s}, A^{(\eta)}(\mathfrak{s})\right) dB_{\mathfrak{s}}. \end{aligned} \tag{3.1.5}$$

Using the inequality $|\sum_{i=1}^3 e_i|^2 \leq 3\sum_{i=1}^3 |e_i|^2$

$$\begin{aligned} & E\left|A^{(\eta+1)}(\mathfrak{x}) - A^{(\eta)}(\mathfrak{x})\right|^2 \\ \leq\ & 3E\left|\int_0^{\mathfrak{x}} \left\{b\left(\mathfrak{s}, A^{(\eta)}(\mathfrak{s})\right) - b\left(\mathfrak{s}, A^{(\eta-1)}(\mathfrak{s})\right)\right\} d\mathfrak{s}\right|^2 \\ & +3E\left|a\int_0^{\mathfrak{x}} (\mathfrak{x}-\mathfrak{s})^{a-1}\left\{\sigma_1\left(\mathfrak{s}, A^{(\eta)}(\mathfrak{s})\right) - \sigma_1\left(\mathfrak{s}, A^{(\eta-1)}(\mathfrak{s})\right)\right\} d\mathfrak{s}\right|^2 \\ & +3E\left|\int_0^{\mathfrak{x}} \left\{\sigma_2\left(\mathfrak{s}, A^{(\eta)}(\mathfrak{s})\right) - \sigma_2\left(\mathfrak{s}, A^{(\eta-1)}(\mathfrak{s})\right)\right\} dB_{\mathfrak{s}}\right|^2 \end{aligned}$$

$$\begin{aligned}
&= 3E\left|\int_0^{\mathfrak{x}} \left\{b\left(\mathfrak{s}, A^{(\eta)}(\mathfrak{s})\right) - b\left(\mathfrak{s}, A^{(\eta-1)}(\mathfrak{s})\right)\right\} d\mathfrak{s}\right|^2 \\
&+3E\left|a\int_0^{\mathfrak{x}} (\mathfrak{x}-\mathfrak{s})^{\frac{a-1}{2}}.(\mathfrak{x}-\mathfrak{s})^{\frac{a-1}{2}} \left\{\sigma_1\left(\mathfrak{s}, A^{(\eta)}(\mathfrak{s})\right) - \sigma_1\left(\mathfrak{s}, A^{(\eta-1)}(\mathfrak{s})\right)\right\} d\mathfrak{s}\right|^2 \\
&+3E\left|\int_0^{\mathfrak{x}} \left\{\sigma_2\left(\mathfrak{s}, A^{(\eta)}(\mathfrak{s})\right) - \sigma_2\left(\mathfrak{s}, A^{(\eta-1)}(\mathfrak{s})\right)\right\} dB_{\mathfrak{s}}\right|^2.
\end{aligned}$$

Cauchy Schwartz inequality on the first two integral and Itô's Isometry on the third integral yield the following:

$$\begin{aligned}
E\left|A^{(\eta+1)}(\mathfrak{x}) - A^{(\eta)}(\mathfrak{x})\right|^2 \leq\ & 3\omega E\int_0^{\mathfrak{x}} \left|b\left(\mathfrak{s}, A^{(\eta)}(\mathfrak{s})\right) - b\left(\mathfrak{s}, A^{(\eta-1)}(\mathfrak{s})\right)\right|^2 d\mathfrak{s} \\
& +3a\mathfrak{x}^a E\int_0^{\mathfrak{x}} (\mathfrak{x}-\mathfrak{s})^{a-1}\left|\sigma_1\left(\mathfrak{s}, A^{(\eta)}(\mathfrak{s})\right)\right. \\
& \left.-\sigma_1\left(\mathfrak{s}, A^{(\eta-1)}(\mathfrak{s})\right)\right|^2 d\mathfrak{s} + 3E\int_0^{\mathfrak{x}} \left|\sigma_2\left(\mathfrak{s}, A^{(\eta)}(\mathfrak{s})\right)\right. \\
& \left.-\sigma_2\left(\mathfrak{s}, A^{(\eta-1)}(\mathfrak{s})\right)\right|^2 d\mathfrak{s}. \qquad (3.1.6)
\end{aligned}$$

Application of the Lipschitz codition (3.1.3) yields:

$$\begin{aligned}
& E\left|A^{(\eta+1)}(\mathfrak{x}) - A^{(\eta)}(\mathfrak{x})\right|^2 \\
\leq\ & 3L^2\omega\int_0^{\mathfrak{x}} E\left|A^{(\eta)}(\mathfrak{s}) - A^{(\eta-1)}(\mathfrak{s})\right|^2 d\mathfrak{s} \\
& +3a\mathfrak{x}^a L^2\int_0^{\mathfrak{x}} (\mathfrak{x}-\mathfrak{s})^{a-1} E\left|A^{(\eta)}(\mathfrak{s}) - A^{(\eta-1)}(\mathfrak{s})\right|^2 d\mathfrak{s} \\
& +3L^2\int_0^{\mathfrak{x}} E\left|A^{(\eta)}(\mathfrak{s}) - A^{(\eta-1)}(\mathfrak{s})\right|^2 d\mathfrak{s} \\
\leq\ & 3L^2(1+\omega)\int_0^{\mathfrak{x}} E\left|A^{(\eta)}(\mathfrak{s}) - A^{(\eta-1)}(\mathfrak{s})\right|^2 d\mathfrak{s} \\
& +3L^2(1+\omega)\int_0^{\mathfrak{x}} (\mathfrak{x}-\mathfrak{s})^{a-1} E\left|A^{(\eta)}(\mathfrak{s}) - A^{(\eta-1)}(\mathfrak{s})\right|^2 d\mathfrak{s}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \mathfrak{x}^{1-a} E\left|A^{(\eta+1)}(\mathfrak{x}) - A^{(\eta)}(\mathfrak{x})\right|^2 \leq\ & 3L^2(1+\omega)\omega^{1-a} \times \left[\omega^{1-a}\int_0^{\mathfrak{x}} \mathfrak{x}^{a-1}\mathfrak{s}^{a-1}\right. \\
& \times\left\{\mathfrak{s}^{1-a} E\left|A^{(\eta)}(\mathfrak{s}) - A^{(\eta-1)}(\mathfrak{s})\right|^2\right\} d\mathfrak{s} \\
& +\int_0^{\mathfrak{x}} (\mathfrak{x}-\mathfrak{s})^{a-1}\mathfrak{s}^{a-1} \times \\
& \left.\left\{\mathfrak{s}^{1-a} E\left|A^{(\eta)}(\mathfrak{s}) - A^{(\eta-1)}(\mathfrak{s})\right|^2\right\} d\mathfrak{s}\right] \quad (3.1.7)
\end{aligned}$$

For locally integrable function $\Psi_1(\mathfrak{x})$ define an operator $\mathfrak{C}_1$ as follows:

$$\mathfrak{C}_1\Psi_1(\mathfrak{x}) := 3L^2(1+\omega)\omega^{1-a}\left[\omega^{1-a}\int_0^{\mathfrak{x}} \mathfrak{x}^{a-1}\mathfrak{s}^{a-1}\Psi_1(\mathfrak{s})d\mathfrak{s} + \int_0^{\mathfrak{x}} (\mathfrak{x}-\mathfrak{s})^{a-1}\mathfrak{s}^{a-1}\Psi_1(\mathfrak{s})d\mathfrak{s}\right] \qquad (3.1.8)$$

From (3.1.7) and (3.1.8), repeating iteration yields:

$$\begin{aligned}\mathfrak{x}^{1-a}E\left|A^{(\eta+1)}(\mathfrak{x})-A^{(\eta)}(\mathfrak{x})\right|^2 &\leq \mathfrak{C}_1\left(\mathfrak{x}^{1-a}E\left|A^{(\eta)}(\mathfrak{x})-A^{(\eta-1)}(\mathfrak{x})\right|^2\right)\\ &\leq \ldots \leq \mathfrak{C}_1^{\eta-1}\left(\mathfrak{x}^{1-a}E\left|A^{(2)}(\mathfrak{x})-A^{(1)}(\mathfrak{x})\right|^2\right)\\ &\leq \mathfrak{C}_1^{\eta}\left(\mathfrak{x}^{1-a}E\left|A^{(1)}(\mathfrak{x})-A^{(0)}(\mathfrak{x})\right|^2\right) \qquad (3.1.9)\end{aligned}$$

As, $E\left|A^{(1)}(\mathfrak{x})-A^{(0)}(\mathfrak{x})\right|^2$ is locally integrable therefore application of (2.1.4), (2.1.5) and (3.1.9) yield:

$$\begin{aligned}\mathfrak{x}^{1-a}E\left|A^{(\eta+1)}(\mathfrak{x})-A^{(\eta)}(\mathfrak{x})\right|^2 &\leq \mathfrak{C}_1^{\eta}\left(\mathfrak{x}^{1-a}E\left|A^{(1)}(\mathfrak{x})-A^{(0)}(\mathfrak{x})\right|^2\right)\\ &\leq (\Gamma(a))^{\eta-1}\prod_{i=1}^{\eta-1}\frac{\Gamma(i(2a-1))}{\Gamma(i(2a-1)+a)}\omega^{-a}\\ &\quad\times\left[3L^2(1+\omega)\omega^a(1+\omega^{1-a})\right]^{\eta}\\ &\quad\times\int_0^{\mathfrak{x}}E\left|A^{(1)}(\mathfrak{s})-A^{(0)}(\mathfrak{s})\right|^2 d\mathfrak{s}. \qquad (3.1.10)\end{aligned}$$

Again, from (3.1.5) applications of the inequality $|\sum_{i=1}^{3}e_i|^2 \leq 3\sum_{i=1}^{3}|e_i|^2$

$$\begin{aligned}&E\left|A^{(1)}(\mathfrak{x})-A^{(0)}(\mathfrak{x})\right|^2\\ &\leq 3E\left|\int_0^{\mathfrak{x}}b(\mathfrak{s},A_0)d\mathfrak{s}\right|^2+3E\left|a\int_0^{\mathfrak{x}}(\mathfrak{x}-\mathfrak{s})^{a-1}\sigma_1(\mathfrak{s},A_0)d\mathfrak{s}\right|^2\\ &+3E\left|\int_0^{\mathfrak{x}}\sigma_2(\mathfrak{s},A_0)dB_{\mathfrak{s}}\right|^2.\end{aligned}$$

Cauchy Schwartz inequality on the first two integral and Itô's Isometry on the third integral yields

$$\begin{aligned}&E\left|A^{(1)}(\mathfrak{x})-A^{(0)}(\mathfrak{x})\right|^2\\ &\leq 3\omega E\int_0^{\mathfrak{x}}|b(\mathfrak{s},A_0)|^2d\mathfrak{s}+3a\mathfrak{x}^aE\int_0^{\mathfrak{x}}(\mathfrak{x}-\mathfrak{s})^{a-1}|\sigma_1(\mathfrak{s},A_0)|^2d\mathfrak{s}\\ &+3E\int_0^{\mathfrak{x}}|\sigma_2(\mathfrak{s},A_0)|^2d\mathfrak{s}\\ &\leq 3\omega E\int_0^{\mathfrak{x}}|b(\mathfrak{s},A_0)|^2d\mathfrak{s}+3a(1+\omega)E\int_0^{\mathfrak{x}}(\mathfrak{x}-\mathfrak{s})^{a-1}|\sigma_1(\mathfrak{s},A_0)|^2d\mathfrak{s}\\ &+3E\int_0^{\mathfrak{x}}|\sigma_2(\mathfrak{s},A_0)|^2d\mathfrak{s}\end{aligned}$$

Linear growth condition yields

$$E\left|A^{(1)}(\mathfrak{x})-A^{(0)}(\mathfrak{x})\right|^2 \leq 3K^2\left(1+E|w_0|^2\right)(1+\omega)(\mathfrak{x}+\mathfrak{x}^a) \qquad (3.1.11)$$

Combination of (3.1.10) and (3.1.11) produces

$$\sup_{0\leq\mathfrak{x}\leq\omega}E\left|A^{(\eta+1)}(\mathfrak{x})-A^{(\eta)}(\mathfrak{x})\right|^2 \leq M_0\prod_{i=1}^{\eta-1}\frac{\Gamma(i(2a-1))}{\Gamma(i(2a-1)+a)}$$

$$\left[3L^2\Gamma(a)\omega^a(1+\omega)(1+\omega^{1-a})\right]^\eta, \tag{3.1.12}$$

provided that

$$M_0 := \frac{3K^2(1+E|w_0)|^2)(1+\omega)}{\Gamma(a)}\left(\frac{\omega}{2}+\frac{\omega^a}{a+1}\right),$$

Thus, for any $\phi,\theta \in N$ such that $\phi > \theta > 0$,

$$\begin{aligned}
\left\|A^{(\phi)}(\mathfrak{x}) - A^{(\theta)}(\mathfrak{x})\right\|^2_{L^2(\mathbb{P})} &\leq \sum_{\eta=\theta}^{\phi}\left\|A^{(\eta+1)}(\mathfrak{x}) - A^{(\eta)}(\mathfrak{x})\right\|^2_{L^2(\mathbb{P})} \\
&= \sum_{\eta=\theta}^{\phi}\int_0^\omega E\left|A^{(\eta+1)}(\mathfrak{x}) - A^{(\eta)}(\mathfrak{x})\right|^2 d\mathfrak{x} \\
&\leq M_1\sum_{\eta=\theta}^{\phi}\left[3L^2\Gamma(a)\omega^a(1+\omega)(1+\omega^{1-a})\right]^\eta \\
&\quad \times\prod_{i=1}^{\eta-1}\frac{\Gamma(i(2a-1))}{\Gamma(i(2a-1)+a)} \to 0,
\end{aligned}$$

for sufficiently large ϕ,θ such that:

$$M_1 := \frac{3K^2(1+E|w_0)|^2(1+\omega)}{\Gamma(a)}\left(\frac{\omega^2}{2(2+a)}+\frac{\omega^{a+1}}{(a+1)(2a+1)}\right),$$

From Doob's maximal inequality for martingales,

$$\begin{aligned}
&\sum_{\eta=1}^{\infty}\mathbb{P}\left[\sup_{0\leq\mathfrak{x}\leq\omega}\left|A^{(\eta+1)}(\mathfrak{x}) - A^{(\eta)}(\mathfrak{x})\right| > \frac{1}{\eta^2}\right] \\
&\leq M_0\sum_{\eta=1}^{\infty}\left[3L^2\Gamma(a)\omega^a(1+\omega)(1+\omega^{1-a})\right]^\eta \\
&\times\prod_{i=1}^{\eta-1}\frac{\Gamma(i(2a-1))}{\Gamma(i(2a-1)+a)}\ \eta^4 < +\infty
\end{aligned}$$

The Borel cantelli lemma yields:

$$\mathbb{P}\left\{\sup_{0\leq\mathfrak{x}\leq\omega}\left|A^{(\eta+1)}(\mathfrak{x}) - A^{(\eta)}(\mathfrak{x})\right| > \frac{1}{\eta^2} \text{ for infinitely many } \eta\right\} = 0,$$

so there exist a random variable $A(\mathfrak{x})$ which is almost surely uniformly continuous on $[0,\omega]$, such that:

$$A^{(\eta)}(\mathfrak{x}) = A^{(0)}(\mathfrak{x}) + \sum_{\theta=0}^{\eta-1}\left(A^{(\theta+1)}(\mathfrak{x}) - A^{(\theta)}(\mathfrak{x})\right) \overset{\eta\to\infty}{\longrightarrow} A(\mathfrak{x}).$$

Since $A^{(\eta)}(\mathfrak{x})$ is $\mathfrak{x}-$continuous for any $\eta \in N$, so $A(\mathfrak{x})$ is also $\mathfrak{x}-$continuous. Therefore,

$$w_0 + \int_0^{\mathfrak{x}} b\left(\mathfrak{s}, A^{(\eta)}(\mathfrak{s})\right) d\mathfrak{s} + a\int_0^{\mathfrak{x}}(\mathfrak{x}-\mathfrak{s})^{a-1}\sigma_1\left(\mathfrak{s}, A^{(\eta)}(\mathfrak{s})\right) d\mathfrak{s}$$

$$+\int_0^{\mathfrak{x}} \sigma_2\left(\mathfrak{s}, A^{(\eta)}(\mathfrak{s})\right) dB_{\mathfrak{s}} \overset{\eta\to\infty}{\longrightarrow} A(\mathfrak{x}),$$

for a stochastic process $A(\mathfrak{x})$ satisfying (3.1.4). □

Theorem 3.1.2 [*22*] *Under the conditions of Theorem* 3.1.1, *stochastic integral equation* (3.1.4) *has at most one solution.*

Proof. Let $A_1(\mathfrak{x})$ and $A_2(\mathfrak{x})$ be solutions of stochastic integral equation (3.1.4), which have the initial conditions $A_i^{(0)}(\mathfrak{x}) = \mathfrak{x}_i$, $1 \leq i \leq 2$. Application of Cauchy-Schwartz inequality, the Itô Isometry, and Lipschitz condition, yield

$$\begin{aligned} E\left|A_1(\mathfrak{x}) - A_2(\mathfrak{x})\right|^2 \leq\ & 4E\left|\mathfrak{x}_1 - \mathfrak{x}_2\right|^2 + 4L^2(1+\omega)\int_0^{\mathfrak{x}} E\left|A_1(\mathfrak{s}) - A_2(\mathfrak{s})\right|^2 d\mathfrak{s} \\ & +4aL^2\omega^a \int_0^{\mathfrak{x}} (\mathfrak{x}-\mathfrak{s})^{a-1} E\left|A_1(\mathfrak{s}) - A_2(\mathfrak{s})\right|^2 d\mathfrak{s} \end{aligned}$$

which can also be written as:

$$\begin{aligned} E|A_1(\mathfrak{x}) - A_2(\mathfrak{x})|^2 \leq\ & 4E|\mathfrak{x}_1 - \mathfrak{x}_2|^2 + 4L^2(1+\omega)\omega^{1-a} \\ & \int_0^{\mathfrak{x}} \mathfrak{x}^{a-1}\mathfrak{s}^{a-1}\left\{\mathfrak{s}^{1-a}E|A_1(\mathfrak{s}) - A_2(\mathfrak{s})|^2\right\} d\mathfrak{s} \\ & +4aL^2\omega^a \int_0^{\mathfrak{x}} (\mathfrak{x}-\mathfrak{s})^{a-1}\mathfrak{s}^{a-1}\left\{\mathfrak{s}^{1-a}E|A_1(\mathfrak{s}) - A_2(\mathfrak{s})|^2\right\} d\mathfrak{s} \end{aligned}$$

Application of Corollary 2.1.2 yields:

$$\begin{aligned} & E|A_1(\mathfrak{x}) - A_2(\mathfrak{x})|^2 \leq 4E|\mathfrak{x}_1 - \mathfrak{x}_2|^2 \times \\ & \qquad F_{2a-1,a-1,2a-1}\left(4L^2\Gamma(a)\left\{(1+\omega)\omega^{1-a} + a\omega^a\right\}\mathfrak{x}^{2a-1}\right). \end{aligned}$$

Since, $A_1(\mathfrak{x})$ and $A_2(\mathfrak{x})$ are solutions of stochstic integral equation (3.1.4), with the initial conditions $A_i^{(0)}(\mathfrak{x}) = \mathfrak{x}_i$, $1 \leq i \leq 2$ therefore $\mathfrak{x}_1 = \mathfrak{x}_2$ and hence

$$E|A_1(\mathfrak{x}) - A_2(\mathfrak{x})|^2 = 0 \text{ for all } \mathfrak{x} > 0,$$

which proves the uniqueness. □

3.2 Delay type differential equations

Example 3.2.1 [*23*] *Consider the following integro-differential equation with several arguments.*

$$[u(\mathfrak{x}_1, \mathfrak{x}_2)]^{\Delta\mathfrak{x}_1\Delta\mathfrak{x}_2} = \mathrm{F}[\mathfrak{x}_1, \mathfrak{t}_1, \mathfrak{x}_2, \mathfrak{t}_2, u(\mu_{11}(\mathfrak{t}_1), \mu_{21}(\mathfrak{t}_2)), \cdots, u(\mu_{1n}(\mathfrak{t}_1), \mu_{2n}(\mathfrak{t}_2)),$$

$$\int_{\mathfrak{x}_{01}}^{\mathfrak{t}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{t}_2}\mathbf{Q}(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2,u(\mu_{11}(\mathfrak{m}_1),\mu_{21}(\mathfrak{m}_2)),\cdots,u(\mu_{1n}(\mathfrak{m}_1),\mu_{2n}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1], \tag{3.2.1}$$

with initial condition

$$\begin{cases} [u(\mathfrak{x}_1,\mathfrak{x}_{02})]^{\Delta\mathfrak{x}_2}=\mathfrak{a}_1^{\Delta}(\mathfrak{x}_1), \quad u(\mathfrak{x}_{01},\mathfrak{x}_2)=\mathfrak{a}_2(\mathfrak{x}_2); & \\ u(\mathfrak{x}_1,\mathfrak{x}_2)=\mathfrak{a}(\mathfrak{x}_1,\mathfrak{x}_2), & \mathfrak{x}_1\in[\mathfrak{p}_1,\mathfrak{x}_{01}]_{\mathbb{T}} \text{ or } \mathfrak{x}_2\in[\mathfrak{p}_2,\mathfrak{x}_{02}]_{\mathbb{T}}; \\ |\mathfrak{a}(\mu_{1i}(\mathfrak{x}_1),\mu_{2i}(\mathfrak{x}_2))|\le|a_1(\mathfrak{x}_1,\mathfrak{x}_2)|, & \mu_{1i}(\mathfrak{x}_1)\le\mathfrak{x}_{01} \text{ or } \mu_{2i}(\mathfrak{x}_2)\le\mathfrak{x}_{02}, \end{cases} \tag{3.2.2}$$

for $\mathrm{F}:\mathbb{T}_1^2\times\mathbb{T}_2^2\times\mathbf{R}^{n+1}\to\mathbf{R}$ *is right-dense continuous on* $\mathbb{T}_1^2\times\mathbb{T}_2^2$ *and continuous on* $\mathbf{R}^{n+1}$; $\mathbf{Q}:\mathbb{T}_1^2\times\mathbb{T}_2^2\times\mathbf{R}^n\to\mathbf{R}$ *is right-dense continuous on* $\mathbb{T}_1^2\times\mathbb{T}_2^2$ *and continuous on* R^n; $u:\mathbb{T}_1\times\mathbb{T}_2\to\mathbf{R}\backslash\{0\}$, $\mathfrak{a}_j:\mathbb{T}_j\to\mathbf{R}$, $\mathfrak{a}:([\mathfrak{p}_1,\mathfrak{x}_{01}]\times[\mathfrak{p}_2,\mathfrak{x}_{02}])_{\mathbb{T}^2}\to\mathbf{R}$, $a_1:\mathbb{T}_1\times\mathbb{T}_2\to\mathbf{R}$ *are right-dense continuous functions and* μ_{ji} *is as defined in Theorem* 2.2.1.

Theorem 3.2.2 *[23] Assume that*

$$\left.\begin{array}{l} |\mathrm{F}(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2,\mathfrak{k}_1,\mathfrak{k}_2\cdots,\mathfrak{k}_n,k)|\le a_2(\mathfrak{x}_1,\mathfrak{x}_2)\sum_{i=1}^n w_1(|\mathfrak{k}_i|) \\ \times[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_2(|\mathfrak{k}_i|)+|k|\}+r_i(\mathfrak{t}_1,\mathfrak{t}_2)]; \\ |\mathbf{Q}(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2,\mathfrak{k}_1\cdots,\mathfrak{k}_n)|\le g_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)w_2(|\mathfrak{k}_i|), \end{array}\right\} \tag{3.2.3}$$

where f_i, g_i, r_i, *and* a_2 *are as defined in* $(C1)-(C2)$; $a_1(\mathfrak{x}_1,\mathfrak{x}_2):=\sum_{j=1}^2\mathfrak{a}_j(\mathfrak{x}_j)$, $w_1(\eta):=\sqrt[3]{\sigma^2(\eta)}+\sqrt[3]{\sigma(\eta)\eta}+\sqrt[3]{\eta^2}$, $w_2(\eta):=\sqrt{\sigma(\sqrt[3]{\eta})}+\sqrt[6]{\eta}$ *for* $\eta\in\mathbf{R}_0^+$. *If* $u(\mathfrak{x}_1,\mathfrak{x}_2)$ *is a solution of the equation* (3.2.1) *satisfying the initial condition* (3.2.2), *then*

$$|u(\mathfrak{x}_1,\mathfrak{x}_2)|\le(\sqrt{\mathfrak{b}_3(\mathfrak{x}_1,\mathfrak{x}_2)}+a_2(\mathfrak{x}_1,\mathfrak{x}_2)\mathfrak{b}_4(\mathfrak{x}_1,\mathfrak{x}_2))^6$$

provided that

$$\mathfrak{b}_3(\mathfrak{x}_1,\mathfrak{x}_2) := \sqrt[3]{|a_1(\mathfrak{x}_1,\mathfrak{x}_2)|}+a_2(\mathfrak{x}_1,\mathfrak{x}_2)\sum_{i=1}^n\int_{\mathfrak{x}_{01}}^{\mathfrak{x}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{x}_2}r_i(\mathfrak{t}_1,\mathfrak{t}_2)\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1.$$

$$\mathfrak{b}_4(\mathfrak{x}_1,\mathfrak{x}_2):=\sum_{i=1}^n\int_{\mathfrak{x}_{01}}^{\mathfrak{x}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{x}_2}f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2) \times(1+\int_{\mathfrak{x}_{01}}^{\mathfrak{t}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{t}_2}g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1)\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1. \tag{3.2.4}$$

Proof. Let $\tilde{\mathbb{T}}:=\varrho(\mathbb{T}_1,\mathfrak{x}_2)$ where $\varrho(\mathfrak{x}_1,\mathfrak{x}_2)$ is strictly increasing for $\mathfrak{x}_1\in\mathbb{T}_1$ and $\tilde{\mathfrak{G}}_j(\eta)=\sqrt[4-j]{\eta}$, then by Theorem 1.2.11, we have

$$[\tilde{\mathfrak{G}}_1(\varrho(\mathfrak{x}_1,\mathfrak{x}_2))]^{\Delta\mathfrak{x}_1} = \tilde{\mathfrak{G}}_1^{\tilde{\Delta}}(\varrho)\varrho^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)$$

$$= \frac{\varrho^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{\sqrt[3]{\sigma^2(\varrho(\mathfrak{x}_1,\mathfrak{x}_2))}+\sqrt[3]{\sigma(\varrho(\mathfrak{x}_1,\mathfrak{x}_2))\varrho(\mathfrak{x}_1,\mathfrak{x}_2)}+\sqrt[3]{\varrho^2(\mathfrak{x}_1,\mathfrak{x}_2)}}$$
$$= \frac{\varrho^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{w_1(\varrho(\mathfrak{x}_1,\mathfrak{x}_2))} = \frac{\varrho^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{w_1(w^{-1}(\varrho(\mathfrak{x}_1,\mathfrak{x}_2)))}.$$

$$[\tilde{\mathfrak{G}}_2(\varrho(\mathfrak{x}_1,\mathfrak{x}_2))]^{\Delta\mathfrak{x}_1} = \tilde{\mathfrak{G}}_2^{\tilde{\Delta}}(\varrho)\varrho^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) = \frac{\varrho^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{\sqrt{\sigma(\varrho(\mathfrak{x}_1,\mathfrak{x}_2))}+\sqrt{\varrho(\mathfrak{x}_1,\mathfrak{x}_2)}}$$
$$= \frac{\varrho^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{w_2(\varrho^3(\mathfrak{x}_1,\mathfrak{x}_2))} = \frac{\varrho^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{w_2(w^{-1}(\mathfrak{G}_1^{-1}(\varrho(\mathfrak{x}_1,\mathfrak{x}_2))))}.$$

The equivalent integral form of (3.2.1) for (3.2.2) is

$$u(\mathfrak{x}_1,\mathfrak{x}_2) = a_1(\mathfrak{x}_1,\mathfrak{x}_2) + \int_{\mathfrak{x}_{01}}^{\mathfrak{x}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{x}_2} \mathrm{F}[\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2,u(\mu_{11}(\mathfrak{t}_1),\mu_{21}(\mathfrak{t}_2)),\cdots,$$
$$u(\mu_{1n}(\mathfrak{t}_1),\mu_{2n}(\mathfrak{t}_2)),\int_{\mathfrak{x}_{01}}^{\mathfrak{t}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{t}_2}\mathbf{Q}(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2,u(\mu_{11}(\mathfrak{m}_1),\mu_{21}(\mathfrak{m}_2)),\cdots,$$
$$u(\mu_{1n}(\mathfrak{m}_1),\mu_{2n}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1 \quad (3.2.5)$$

Use of modulus and (3.2.3), equation (3.2.5) has the form

$$\begin{aligned}
&|u(\mathfrak{x}_1,\mathfrak{x}_2)|\\
\le\ & |a_1(\mathfrak{x}_1,\mathfrak{x}_2)| + \int_{\mathfrak{x}_{01}}^{\mathfrak{x}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{x}_2}|\mathrm{F}[\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2,u(\mu_{11}(\mathfrak{t}_1),\mu_{21}(\mathfrak{t}_2)),\cdots,u(\mu_{1n}(\mathfrak{t}_1),\mu_{2n}(\mathfrak{t}_2)),\\
&\int_{\mathfrak{x}_{01}}^{\mathfrak{t}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{t}_2}\mathbf{Q}(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2,u(\mu_{11}(\mathfrak{m}_1),\mu_{21}(\mathfrak{m}_2)),\cdots,\\
&u(\mu_{1n}(\mathfrak{m}_1),\mu_{2n}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1]|\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1\\
\le\ & |a_1(\mathfrak{x}_1,\mathfrak{x}_2)| + a_2(\mathfrak{x}_1,\mathfrak{x}_2)\sum_{i=1}^{n}\int_{\mathfrak{x}_{01}}^{\mathfrak{x}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{x}_2} w_1(|u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2))|)[\mathfrak{f}_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\\
&\times\{w_2(|u(\mu_{1i}(\mathfrak{t}_1),\mu_{2i}(\mathfrak{t}_2))|) + \int_{\mathfrak{x}_{01}}^{\mathfrak{t}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2)\\
&\times w_2(|u(\mu_{1i}(\mathfrak{m}_1),\mu_{2i}(\mathfrak{m}_2))|)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\} + r_i(\mathfrak{t}_1,\mathfrak{t}_2)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1, \qquad (3.2.6)
\end{aligned}$$

here an immediate application of inequality (2.2.5) to (3.2.6) yields the desired result.

□

Example 3.2.3 [23] *Consider the following integro-differential equation with several arguments.*

$$\begin{aligned}
&[u^{\frac{2}{5}}(\mathfrak{x}_1,\mathfrak{x}_2)u^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)]^{\Delta\mathfrak{x}_2}\\
&= \mathrm{F}[\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2,u(\mu_{11}(\mathfrak{t}_1),\mu_{21}(\mathfrak{t}_2)),\cdots,u(\mu_{1n}(\mathfrak{t}_1),\mu_{2n}(\mathfrak{t}_2)), \qquad (3.2.7)\\
&\int_{\mathfrak{x}_{01}}^{\mathfrak{t}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{t}_2}\mathbf{Q}(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2,u(\mu_{11}(\mathfrak{m}_1),\mu_{21}(\mathfrak{m}_2)),\cdots,u(\mu_{1n}(\mathfrak{m}_1),\mu_{2n}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1]
\end{aligned}$$

with initial condition

$$\begin{cases} u^{\frac{2}{5}}(\mathfrak{x}_1,\mathfrak{x}_{02})u^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_{02}) = \frac{5\mathfrak{f}_1^{\Delta}(\mathfrak{x}_1)}{3};\ u^{\frac{3}{5}}(\mathfrak{x}_{01},\mathfrak{x}_2) = \mathfrak{f}_2(\mathfrak{x}_2), & \\ u(\mathfrak{x}_1,\mathfrak{x}_2) = \mathfrak{a}(\mathfrak{x}_1,\mathfrak{x}_2), & \mathfrak{x}_1 \in [\mathfrak{p}_1,\mathfrak{x}_{01}]_{\mathbb{T}} \text{ or } \mathfrak{x}_2 \in [\mathfrak{p}_2,\mathfrak{x}_{02}]_{\mathbb{T}}\ ; \\ |\mathfrak{a}(\mu_{1i}(\mathfrak{x}_1),\mu_{2i}(\mathfrak{x}_2))| \le \mathfrak{C}^{\frac{1}{5}}, & \mu_{1i}(\mathfrak{x}_1) \le \mathfrak{x}_{01} \text{ or } \mu_{2i}(\mathfrak{x}_2) \le \mathfrak{x}_{02}, \end{cases} \tag{3.2.8}$$

for $\mathfrak{f}_j : \mathbb{T}_j \to \mathbf{R}$, $\mathfrak{C}$ *is a non zero constant such that* $\mathfrak{C} \ge \sum_{j=1}^{2} |\mathfrak{f}_j(x_j)|$. *Where* F, u, $\mathfrak{a}$, $\mathbf{Q}$ *and* μ_{ji} *is as defined in Theorem* 3.2.2.

Theorem 3.2.4 *[23] Assume that*

$$\left.\begin{aligned} &|\mathrm{F}(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2,\mathfrak{k}_1,\mathfrak{k}_2\cdots,\mathfrak{k}_n,k)| \\ &\le \textstyle\sum_{i=1}^{n} \frac{|\mathfrak{k}_i|^2}{3}[f_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)\{w_2(|\mathfrak{k}_i|)+|k|\} + r_i(\mathfrak{t}_1,\mathfrak{t}_2)], \\ &|\mathbf{Q}(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2,\mathfrak{k}_1\cdots,\mathfrak{k}_n)| \le g_i(\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2)w_2(|\mathfrak{k}_i|). \end{aligned}\right\} \tag{3.2.9}$$

where f_i, g_i, *and* r_i *are as defined in* $(C1)-(C2)$; $w_2(\eta) = \sqrt[3]{\sigma^2(\eta^3)} + \sqrt[3]{\sigma(\eta^3)\eta^3} + \eta^2$, *for* $\eta \in \mathbf{R}_0^+$. *If* $u(\mathfrak{x}_1,\mathfrak{x}_2)$ *is a solution of the equation* (3.2.7) *satisfying the initial condition* (3.2.8), *then*

$$|u(\mathfrak{x}_1,\mathfrak{x}_2)| \le \sqrt[3]{\overline{\mathfrak{b}}_3(\mathfrak{x}_1,\mathfrak{x}_2)} + \mathfrak{b}_4(\mathfrak{x}_1,\mathfrak{x}_2),$$

provided that $\mathfrak{b}_4(\mathfrak{x}_1,\mathfrak{x}_2)$ *is defined by* (3.2.4) *and*

$$\overline{\mathfrak{b}}_3(\mathfrak{x}_1,\mathfrak{x}_2) := \mathfrak{C}^{\frac{3}{5}} + \sum_{i=1}^{n} \int_{\mathfrak{x}_{01}}^{\mathfrak{x}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{x}_2} r_i(\mathfrak{t}_1,\mathfrak{t}_2)\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1.$$

Proof. Let $\overline{\mathbb{T}} := \varpi(\mathbb{T}_1,\mathfrak{x}_2)$ where ϖ is strictly increasing for $\mathfrak{x}_1 \in \mathbb{T}_1$ and $\overline{\mathfrak{G}}_2(\eta) = \sqrt[3]{\eta}$, then by Theorem 1.2.11, we have

$$\begin{aligned} [\overline{\mathfrak{G}}_2(\varpi(\mathfrak{x}_1,\mathfrak{x}_2))]^{\Delta\mathfrak{x}_1} &= \overline{\mathfrak{G}}_2^{\Delta}(\varpi)\varpi^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \\ &= \frac{\varpi^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{\sqrt[3]{\sigma^2(\varpi(\mathfrak{x}_1,\mathfrak{x}_2))} + \sqrt[3]{\sigma(\varpi(\mathfrak{x}_1,\mathfrak{x}_2))\varpi(\mathfrak{x}_1,\mathfrak{x}_2)} + \sqrt[3]{\varpi^2(\mathfrak{x}_1,\mathfrak{x}_2)}} \\ &= \frac{\varpi^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{w_2(\sqrt[3]{\varpi(\mathfrak{x}_1,\mathfrak{x}_2)})} \end{aligned}$$

By Integrating equation (3.2.7) over $[\mathfrak{x}_{02},\mathfrak{x}_2]$, we have

$$\begin{aligned} &u^{\frac{2}{5}}(\mathfrak{x}_1,\mathfrak{x}_2)u^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2) \\ &= u^{\frac{2}{5}}(\mathfrak{x}_1,\mathfrak{x}_{02})u^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_{02}) + \int_{\mathfrak{x}_{02}}^{\mathfrak{x}_2} \mathrm{F}[\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2,u(\mu_{11}(\mathfrak{t}_1),\mu_{21}(\mathfrak{t}_2)),\cdots, \\ &u(\mu_{1n}(\mathfrak{t}_1),\mu_{2n}(\mathfrak{t}_2)), \int_{\mathfrak{x}_{01}}^{\mathfrak{t}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{t}_2} \mathbf{Q}(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2,u(\mu_{11}(\mathfrak{m}_1),\mu_{21}(\mathfrak{m}_2)),\cdots, \end{aligned}$$

$$u(\mu_{1n}(\mathfrak{m}_1),\mu_{2n}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1]\Delta\mathfrak{t}_2 \tag{3.2.10}$$

By Theorem 1.2.9, we have

$$\begin{aligned}
(\frac{5}{3}u^{\frac{3}{5}}(\mathfrak{x}_1,\mathfrak{x}_2))^{\Delta\mathfrak{x}_1} &= u^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)\int_0^1\{u(\mathfrak{x}_1,\mathfrak{x}_2)+h\mu(\mathfrak{x}_1,\mathfrak{x}_2)u^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)\}^{-\frac{2}{5}}dh\\
&= \frac{u^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{u^{\frac{2}{5}}(\mathfrak{x}_1,\mathfrak{x}_2)}\int_0^1\{1+h\mu(\mathfrak{x}_1,\mathfrak{x}_2)\frac{u^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{u(\mathfrak{x}_1,\mathfrak{x}_2)}\}^{-\frac{2}{5}}dh\\
&= \frac{u^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{u^{\frac{2}{5}}(\mathfrak{x}_1,\mathfrak{x}_2)}\times\left|\frac{5\{1+h\mu(\mathfrak{x}_1,\mathfrak{x}_2)\frac{u^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{u(\mathfrak{x}_1,\mathfrak{x}_2)}\}^{\frac{3}{5}}}{3\mu(\mathfrak{x}_1,\mathfrak{x}_2)\frac{u^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{u(\mathfrak{x}_1,\mathfrak{x}_2)}}\right|_0^1\\
&= \frac{u^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{u^{\frac{2}{5}}(\mathfrak{x}_1,\mathfrak{x}_2)}\times\frac{5\{1+\mu(\mathfrak{x}_1,\mathfrak{x}_2)\frac{u^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{u(\mathfrak{x}_1,\mathfrak{x}_2)}\}^{\frac{3}{5}}-5}{3\mu(\mathfrak{x}_1,\mathfrak{x}_2)\frac{u^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{u(\mathfrak{x}_1,\mathfrak{x}_2)}}
\end{aligned} \tag{3.2.11}$$

By Bernoulli's inequality for $\frac{u^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{u(\mathfrak{x}_1,\mathfrak{x}_2)}\geq 0$, we have

$$(\frac{5}{3}u^{\frac{3}{5}}(\mathfrak{x}_1,\mathfrak{x}_2))^{\Delta\mathfrak{x}_1}\leq\frac{u^{\Delta\mathfrak{x}_1}(\mathfrak{x}_1,\mathfrak{x}_2)}{u^{\frac{2}{5}}(\mathfrak{x}_1,\mathfrak{x}_2)} \tag{3.2.12}$$

From (3.2.10) and (3.2.12), we have

$$\begin{aligned}
&(\frac{5}{3}u^{\frac{3}{5}}(\mathfrak{x}_1,\mathfrak{x}_2))^{\Delta\mathfrak{x}_1}\\
&\leq\frac{5\mathfrak{f}_1^{\Delta}(\mathfrak{x}_1)}{3}+\int_{\mathfrak{x}_{02}}^{\mathfrak{x}_2}\mathrm{F}[\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2,u(\mu_{11}(\mathfrak{t}_1),\mu_{21}(\mathfrak{t}_2)),\cdots,u(\mu_{1n}(\mathfrak{t}_1),\mu_{2n}(\mathfrak{t}_2)),\\
&\int_{\mathfrak{x}_{01}}^{\mathfrak{t}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{t}_2}\mathbf{Q}(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2,u(\mu_{11}(\mathfrak{m}_1),\mu_{21}(\mathfrak{m}_2)),\cdots,\\
&u(\mu_{1n}(\mathfrak{m}_1),\mu_{2n}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1]\Delta\mathfrak{t}_2
\end{aligned}$$

Integrating over $[\mathfrak{x}_{01},\mathfrak{x}_1]$ yields

$$\begin{aligned}
u^{\frac{3}{5}}(\mathfrak{x}_1,\mathfrak{x}_2)\leq\mathfrak{f}_1(\mathfrak{x}_1)+\mathfrak{f}_2(\mathfrak{x}_1)+\frac{3}{5}\int_{\mathfrak{x}_{01}}^{\mathfrak{x}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{x}_2}\mathrm{F}[\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2,u(\mu_{11}(\mathfrak{t}_1),\mu_{21}(\mathfrak{t}_2)),\cdots,\\
u(\mu_{1n}(\mathfrak{t}_1),\mu_{2n}(\mathfrak{t}_2)),\int_{\mathfrak{x}_{01}}^{\mathfrak{t}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{t}_2}\mathbf{Q}(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2,u(\mu_{11}(\mathfrak{m}_1),\mu_{21}(\mathfrak{m}_2)),\cdots,\\
u(\mu_{1n}(\mathfrak{m}_1),\mu_{2n}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1
\end{aligned} \tag{3.2.13}$$

Use of modulus and (3.2.9), inequality (3.2.13) has the form

$$\begin{aligned}
&|u^{\frac{3}{5}}(\mathfrak{x}_1,\mathfrak{x}_2)|\\
\leq\ &\mathfrak{C}+\frac{3}{5}\int_{\mathfrak{x}_{01}}^{\mathfrak{x}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{x}_2}|\mathrm{F}[\mathfrak{x}_1,\mathfrak{t}_1,\mathfrak{x}_2,\mathfrak{t}_2,u(\mu_{11}(\mathfrak{t}_1),\mu_{21}(\mathfrak{t}_2)),\cdots,u(\mu_{1n}(\mathfrak{t}_1),\mu_{2n}(\mathfrak{t}_2)),\\
&\int_{\mathfrak{x}_{01}}^{\mathfrak{t}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{t}_2}\mathbf{Q}(\mathfrak{t}_1,\mathfrak{m}_1,\mathfrak{t}_2,\mathfrak{m}_2,u(\mu_{11}(\mathfrak{m}_1),\mu_{21}(\mathfrak{m}_2)),\cdots,
\end{aligned}$$

$$
\begin{aligned}
& u(\mu_{1n}(\mathfrak{m}_1), \mu_{2n}(\mathfrak{m}_2)))\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1]|\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1 \\
\leq \quad & \mathfrak{C} + \frac{1}{5}\sum_{i=1}^{n}\int_{\mathfrak{x}_{01}}^{\mathfrak{x}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{x}_2} |u(\mu_{1i}(\mathfrak{t}_1), \mu_{2i}(\mathfrak{t}_2))|^2[f_i(\mathfrak{x}_1, \mathfrak{t}_1, \mathfrak{x}_2, \mathfrak{t}_2) \\
& \times\{w_2(|u(\mu_{1i}(\mathfrak{t}_1), \mu_{2i}(\mathfrak{t}_2))|) + \int_{\mathfrak{x}_{01}}^{\mathfrak{t}_1}\int_{\mathfrak{x}_{02}}^{\mathfrak{t}_2} g_i(\mathfrak{t}_1, \mathfrak{m}_1, \mathfrak{t}_2, \mathfrak{m}_2) \\
& \times w_2(|u(\mu_{1i}(\mathfrak{m}_1), \mu_{2i}(\mathfrak{m}_2))|)\Delta\mathfrak{m}_2\Delta\mathfrak{m}_1\} + r_i(\mathfrak{t}_1, \mathfrak{t}_2)]\Delta\mathfrak{t}_2\Delta\mathfrak{t}_1, \qquad (3.2.14)
\end{aligned}
$$

here an immediate application of inequality (2.2.66) to (3.2.14) yields the desired result. ∎

Example 3.2.5 [*23*] *Consider the delay discrete inequality* (2.2.83) *satisfying the initial condition* (2.2.84) *with* $u(\mathfrak{x}_1, \mathfrak{x}_2) = 27^{\mathfrak{x}_1\mathfrak{x}_2}$, $\rho_{ji} = ji$, $a_1(\mathfrak{x}_1, \mathfrak{x}_2) = 27^8$, $a_2(\mathfrak{x}_1, \mathfrak{x}_2) = \sqrt[10]{\mathfrak{x}_1\mathfrak{x}_2}$, $r_i(\mathfrak{x}_1, \mathfrak{x}_2) = \sqrt[i+1]{\exp(\mathfrak{x}_1\mathfrak{x}_2)}$, $f_i(\mathfrak{x}_1, \mathfrak{t}_1, \mathfrak{x}_2, \mathfrak{t}_2) = \arctan(\sqrt[i+1]{\mathfrak{x}_1 + \mathfrak{t}_1 + \mathfrak{x}_2 + \mathfrak{t}_2})$, $\gamma_{ji} \equiv I \equiv w$, $g_i(\mathfrak{x}_1, \mathfrak{t}_1, \mathfrak{x}_2, \mathfrak{t}_2) = 10^{-i-1}\sqrt[i+1]{\mathfrak{x}_1 + \mathfrak{t}_1 + \mathfrak{x}_2 + \mathfrak{t}_2}$, $1 \leq i \leq 2$. $\tilde{\mathfrak{G}}_j$ *and* w_j *are as defined in Theorem* 3.2.2.

We compute the values of $u(\mathfrak{x}_1, \mathfrak{x}_2)$ *from* (2.2.83) *and also we compute the value of* $u(\mathfrak{x}_1, \mathfrak{x}_2)$ *by using the result* (2.2.85)*. In our calculation, we use* (2.2.83) *and* (2.2.85) *as equations. We easily find that numerical solution agrees with the analytical solution for some discrete inequalities.*

$(\mathfrak{x}_1, \mathfrak{x}_2)$	(2.2.83)	(2.2.85)
$(2, 2)$	*3.0692e+11*	*5.7300e+11*
$(2, 5)$	*3.1140e+11*	*1.9384e+12*
$(2, 9)$	*3.1481e+11*	*9.2602e+12*
$(3, 4)$	*3.1193e+11*	*2.8268e+12*
$(3, 8)$	*3.1605e+11*	*3.9592e+13*
$(7, 3)$	*3.1497e+11*	*1.7927e+13*
$(7, 4)$	*3.1834e+11*	*2.9224e+14*
$(11, 5)$	*3.2673e+26*	*6.3683e+26*
$(15, 2)$	*3.1803e+11*	*1.2336e+14*
$(40, 1)$	*3.2160e+11*	*1.9787e+14*

3.3 Fractional Cauchy type problem on time scales

Consider the following Cauchy type problem with Riemann-Liouville fractional derivative

$$\left.\begin{aligned} &D^{a}_{\Delta,\omega_0}\ r(\mathfrak{x}) = \mathfrak{S}(\mathfrak{x}, r(\mathfrak{x}));\\ &D^{a-1}_{\Delta,\omega_0}\ r(\omega_0) = \mathfrak{b} \end{aligned}\right\} \tag{3.3.1}$$

where $a \in (0,1)$.

The following result gives us the estimation of the solution of the Cauchy type initial value problem (3.3.1).

Theorem 3.3.1 *[24] Let $\omega_0, \mathfrak{x} \in \mathbb{T}_1$ and $G \in \mathbf{R}$ an open set. Let $\mathfrak{S} : \mathbb{T}_1 \times G \to \mathbf{R}$ be a function such that $\mathfrak{S}(\mathfrak{x}, r) \in L_\Delta[\omega_0, \omega)$ for any $r \in G$. If $r(\mathfrak{x}) \in L^a_\Delta[\omega_0, \omega)$ such that $|\mathfrak{S}(\mathfrak{x}, r)| \le |r|^b$, $b \in (0,1)$. Then the cauchy type problem* (3.3.1) *has the following explicit bound*

$$|r(\mathfrak{x})| \ \le\ \sum_{\theta=0}^{\infty}\sum_{\vartheta=0}^{\theta}\binom{\theta}{\vartheta}\left(b\xi^{b-1}\right)^{\theta} I^{\vartheta a-\vartheta+\theta}_{\Delta,\omega_0}\ \widetilde{|\mathfrak{b}|h_{a-1}}(\mathfrak{x},\omega_0), \tag{3.3.2}$$

provided that

$$\widetilde{|\mathfrak{b}|h_{a-1}}(\mathfrak{x},\omega_0) \ :=\ |\mathfrak{b}|h_{a-1}(\mathfrak{x},\omega_0) + (1-b)\xi^{b}\left\{(\mathfrak{x}-\omega_0) + h_a(\mathfrak{x},\omega_0)\right\},$$

Here $L_\Delta[\omega_0,\omega) := L_{\Delta,1}[\omega_0,\omega)$ be the space of $\Delta-$Lebesgue integrable functions in a finite interval $[\omega_0,\omega)_{\mathbb{T}}$ and $L^a_\Delta[\omega_0,\omega) := \left\{r \in L_\Delta[\omega_0,\omega) :\ D^a_{\Delta,\omega_0} \in L_\Delta[\omega_0,\omega)\right\}$.

Proof. The equivalent integral form of the initial value problem (3.3.1) is

$$r(\mathfrak{x}) = \mathfrak{b}h_{a-1}(\mathfrak{x},\omega_0) + I^a_{\Delta,\omega_0}\mathfrak{S}(\mathfrak{x}, r(\mathfrak{x})).$$

Then,

$$\begin{aligned} |r(\mathfrak{x})| \ &\le\ |\mathfrak{b}|h_{a-1}(\mathfrak{x},\omega_0) + I^a_{\Delta,\omega_0}\,|\mathfrak{S}(\mathfrak{x}, r(\mathfrak{x}))| \\ &\le\ |\mathfrak{b}|h_{a-1}(\mathfrak{x},\omega_0) + I_{\Delta,\omega_0}\,|r(\mathfrak{x})|^b + I^a_{\Delta,\omega_0}\,|r(\mathfrak{x})|^b \end{aligned} \tag{3.3.3}$$

An application of Theorem 2.3.1 to (3.3.3), yields the desired result. □

Theorem 3.3.2 *[24] Let the conditions of Theorem* 3.3.1 *be satisfied. Moreover, if*

$$|\mathfrak{S}(\mathfrak{x}, r) - \mathfrak{S}(\mathfrak{x}, s)| \le |r-s|^b,$$

then (3.3.1) *has at most one solution.*

Proof. Suppose that initial value problem (3.3.1) has two solutions $r_i(\mathfrak{x}),\ 1 \le i \le 2$. We have

$$r_i(\mathfrak{x}) = \mathfrak{b}h_{a-1}(\mathfrak{x}, \omega_0) + I^a_{\Delta,\omega_0}\mathfrak{S}(\mathfrak{x}, r_i(\mathfrak{x})), \quad 1 \le i \le 2.$$

Hence,

$$r_1(\mathfrak{x}) - r_2(\mathfrak{x}) = I^a_{\Delta,\omega_0}\left[\mathfrak{S}(\mathfrak{x}, r_1(\mathfrak{x})) - \mathfrak{S}(\mathfrak{x}, r_2(\mathfrak{x}))\right]$$

or

$$|r_1(\mathfrak{x}) - r_2(\mathfrak{x})| \le I^a_{\Delta,\omega_0}\left|r_1(\mathfrak{x}) - r_2(\mathfrak{x})\right|^b + I_{\Delta,\omega_0}\left|r_1(\mathfrak{x}) - r_2(\mathfrak{x})\right|^b \tag{3.3.4}$$

Considering $|r_1(\mathfrak{x}) - r_2(\mathfrak{x})|$ as one independent function and applying Theorem 2.3.1 to inequality (3.3.4), we get $|r_1(\mathfrak{x}) - r_2(\mathfrak{x})| \le 0$. Therefor, $r_1(\mathfrak{x}) = r_2(\mathfrak{x})$. ◻

3.4 Fractional Stochastic differential equation on time scales

Consider the following nonlinear fractional $\Delta-$stochastic differential equation

$$\left.\begin{aligned} &\Delta\mathfrak{N}(\mathfrak{q}(\mathfrak{x})) = b\left(\mathfrak{x}, \mathfrak{N}(\mathfrak{q}(\mathfrak{x}))\right)\Delta\mathfrak{x} + \sigma_1\left(\mathfrak{x}, \mathfrak{N}(\mathfrak{q}(\mathfrak{x}))\right)\Delta\mathfrak{x}^a \\ &\qquad\qquad + \sigma_2\left(\mathfrak{x}, \mathfrak{N}(\mathfrak{q}(\mathfrak{x}))\right)\Delta B_{\mathfrak{x}}; \\ &\mathfrak{N}(\mathfrak{q}(\omega_0)) = \mathfrak{N}(\mathfrak{q}_0), \end{aligned}\right\} \tag{3.4.1}$$

where $B_{\mathfrak{x}}$ is the standard Brownian motion.

Theorem 3.4.1 *[24] Let $(\Omega, \mathcal{G}, \rho)$ be a complete probability space with an $m-$dimensional Brownian motion $B(\mathfrak{x}) := (B_1(\mathfrak{x}), ..., B_m(\mathfrak{x}))^T$ defined on the space $\mathbf{R}^n$, $\mathfrak{x} > 0$ and $a \in (0,1)$; let w_0 be a random variable such that $E|w_0|^2 < \infty$. Let $b, \sigma_1 : [0,\omega]_{\mathbb{T}} \times \mathbf{R}^n \to \mathbf{R}^n$ and $\sigma_2 : [0,\omega]_{\mathbb{T}} \times \mathbf{R}^n \to \mathbf{R}^{n\times m}$ be right dense-continuous on $[0,\omega]_{\mathbb{T}}$, continuous on $\mathbf{R}^n$ and measurable. Let $\mathfrak{N} : \mathbf{R}^n \to \mathbf{R}^n$ be continuous on $\mathbf{R}^n$ such that:*

$$\left|b\left(\mathfrak{x}, \mathfrak{N}(\mathfrak{q})\right)\right|^2 + \left|\sigma_1\left(\mathfrak{x}, \mathfrak{N}(\mathfrak{q})\right)\right|^2 + \left|\sigma_2\left(\mathfrak{x}, \mathfrak{N}(\mathfrak{q})\right)\right|^2 \le K^2\left(1 + |\mathfrak{N}(\mathfrak{q})|^2\right) \tag{3.4.2}$$

$$\begin{aligned} &\left|b\left(\mathfrak{x}, \mathfrak{N}(\mathfrak{q})\right) - b\left(\mathfrak{x}, \mathfrak{N}(\mathfrak{y})\right)\right| + \left|\sigma_1\left(\mathfrak{x}, \mathfrak{N}(\mathfrak{q})\right) - \sigma_1\left(\mathfrak{x}, \mathfrak{N}(\mathfrak{y})\right)\right| \\ &\qquad + \left|\sigma_2\left(\mathfrak{x}, \mathfrak{N}(\mathfrak{q})\right) - \sigma_2\left(\mathfrak{x}, \mathfrak{N}(\mathfrak{y})\right)\right| \le \mathrm{L}|\mathfrak{N}(\mathfrak{q}) - \mathfrak{N}(\mathfrak{y})| \end{aligned} \tag{3.4.3}$$

for some constants $K, \mathrm{L} > 0$. *Then the* $\Delta-$*stochastic differential equation* (3.4.1) *has a* $\mathfrak{x}-$*continuous solution with a filtration* $\mathcal{G}_{\mathfrak{x}}^{w_0}$ *such that*

$$E\left[\int_0^{\omega} |\mathfrak{N}(\mathfrak{q})|^2 \Delta\mathfrak{x}\right] < \infty.$$

Proof. The integral form of the $\Delta-$stochastic differential equation (3.4.1) is as follows:

$$\begin{aligned}\mathfrak{N}(\mathfrak{q}(\mathfrak{x})) &= \mathfrak{N}(w_0) + I_{\Delta,0}\, b(\mathfrak{x}, \mathfrak{N}(\mathfrak{q}(\mathfrak{x}))) + \Gamma(a+1) I_{\Delta,0}^{a}\, \sigma_1(\mathfrak{x}, \mathfrak{N}(\mathfrak{q}(\mathfrak{x}))) \\ &\quad + \int_0^{\mathfrak{x}} \sigma_2(\mathfrak{s}, \mathfrak{N}(\mathfrak{q}(\mathfrak{s})))\, \Delta B_{\mathfrak{s}} \end{aligned} \tag{3.4.4}$$

By the method of Picard-Lindelöf iteration, define iteratively $\mathfrak{N}(\mathfrak{q}^{(0)}(\mathfrak{x})) = \mathfrak{N}(\omega_0)$, for some $\eta \in \mathbb{N}$, as follows:

$$\begin{aligned}\mathfrak{N}(\mathfrak{q}^{(\eta+1)}(\mathfrak{x})) &= \mathfrak{N}(w_0) + I_{\Delta,0}\, b\left(\mathfrak{x}, \mathfrak{N}(\mathfrak{q}^{(\eta)}(\mathfrak{x}))\right) + \Gamma(a+1) I_{\Delta,0}^{a}\, \sigma_1\left(\mathfrak{x}, \mathfrak{N}(\mathfrak{q}^{(\eta)}(\mathfrak{x}))\right) \\ &\quad + \int_0^{\mathfrak{x}} \sigma_2\left(\mathfrak{s}, \mathfrak{N}(\mathfrak{q}^{(\eta)}(\mathfrak{s}))\right) \Delta B_s. \end{aligned} \tag{3.4.5}$$

Using the inequality $|\sum_{i=1}^{3} e_i|^2 \leq 3\sum_{i=1}^{3} |e_i|^2$, we have

$$\begin{aligned} & E\left|\mathfrak{N}\left(\mathfrak{q}^{(\eta+1)}(\mathfrak{x})\right) - \mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)\right|^2 \\ \leq\ & 3E\left|I_{\Delta,0}\left\{b\left(\mathfrak{x}, \mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)\right) - b\left(\mathfrak{x}, \mathfrak{N}\left(\mathfrak{q}^{(\eta-1)}(\mathfrak{x})\right)\right)\right\}\right|^2 \\ & +3E\left|\Gamma(a+1) I_{\Delta,0}^{a}\left\{\sigma_1\left(\mathfrak{x}, \mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)\right) - \sigma_1\left(\mathfrak{x}, \mathfrak{N}\left(\mathfrak{q}^{(\eta-1)}(\mathfrak{x})\right)\right)\right\}\right|^2 \\ & +3E\left|\int_0^{\mathfrak{x}}\left\{\sigma_2\left(\mathfrak{s}, \mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{s})\right)\right) - \sigma_2\left(\mathfrak{s}, \mathfrak{N}\left(\mathfrak{q}^{(\eta-1)}(\mathfrak{s})\right)\right)\right\}\Delta B_{\mathfrak{s}}\right|^2 \\ =\ & 3E\left|I_{\Delta,0}\left\{b\left(\mathfrak{x}, \mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)\right) - b\left(\mathfrak{x}, \mathfrak{N}\left(\mathfrak{q}^{(\eta-1)}(\mathfrak{x})\right)\right)\right\}\right|^2 \\ & +3E\left|\Gamma(a+1)\int_0^{\mathfrak{x}}(h_{a-1}(\mathfrak{x}, \sigma(\mathfrak{s})))^{\frac{1}{2}}\right. \\ & \left.\times(h_{a-1}(\mathfrak{x}, \sigma(\mathfrak{s})))^{\frac{1}{2}}\left\{\sigma_1\left(\mathfrak{s}, \mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{s})\right)\right) - \sigma_1\left(\mathfrak{s}, \mathfrak{N}\left(\mathfrak{q}^{(\eta-1)}(\mathfrak{s})\right)\right)\right\}\Delta\mathfrak{s}\right|^2 \\ & +3E\left|\int_0^{x}\left\{\sigma_2\left(\mathfrak{s}, \mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{s})\right)\right) - \sigma_2\left(\mathfrak{s}, \mathfrak{N}\left(\mathfrak{q}^{(\eta-1)}(\mathfrak{s})\right)\right)\right\}\Delta B_{\mathfrak{s}}\right|^2. \end{aligned}$$

Cauchy Schwartz inequality on the first two integral and Itô's Isometry on the third integral yield the following:

$$\begin{aligned} & E\left|\mathfrak{N}\left(\mathfrak{q}^{(\eta+1)}(\mathfrak{x})\right) - \mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)\right|^2 \\ \leq\ & 3\mathfrak{x} E I_{\Delta,0}\left[b\left(\mathfrak{x}, \mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)\right) - b\left(\mathfrak{x}, \mathfrak{N}\left(\mathfrak{q}^{(\eta-1)}(\mathfrak{x})\right)\right)\right]^2 \\ & +3\left(\Gamma(a+1)\right)^2 h_a(\mathfrak{x}, 0) E I_{\Delta,0}^{a}\left[\sigma_1\left(\mathfrak{x}, \mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)\right) - \sigma_1\left(\mathfrak{x}, \mathfrak{N}\left(\mathfrak{q}^{(\eta-1)}(\mathfrak{x})\right)\right)\right]^2 \end{aligned}$$

$$+3EI_{\Delta,0}\left[\sigma_2\left(\mathfrak{x},\mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)\right)-\sigma_2\left(\mathfrak{x},\mathfrak{N}\left(\mathfrak{q}^{(\eta-1)}(\mathfrak{x})\right)\right)\right]^2. \tag{3.4.6}$$

An application of the Lipschitz condition (3.4.3), yields:

$$\begin{aligned}
& E\left|\mathfrak{N}\left(\mathfrak{q}^{(\eta+1)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)\right|^2 \\
\leq\ & 3\mathrm{L}^2(\mathfrak{x}+1)I_{\Delta,0}\left\{E\left|\mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(\eta-1)}(\mathfrak{x})\right)\right|^2\right\} \\
& +3\mathrm{L}^2\left(\Gamma(a+1)\right)^2 h_a(\mathfrak{x},0)I^a_{\Delta,0}\left\{E\left|\mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(\eta-1)}(\mathfrak{x})\right)\right|^2\right\}. \qquad (3.4.7)
\end{aligned}$$

For continuous function $\Psi_2(\mathfrak{x})$, we define an operator $\mathfrak{C}_2$ as follows:

$$\mathfrak{C}_2\Psi_2(\mathfrak{x}):=3\mathrm{L}^2(\mathfrak{x}+1)I_{\Delta,0}\Psi_2(\mathfrak{x})+3\mathrm{L}^2\left(\Gamma(a+1)\right)^2 h_a(\mathfrak{x},0)I^a_{\Delta,0}\Psi_2(\mathfrak{x}) \tag{3.4.8}$$

Repeated iterations on (3.4.7) for (3.4.8), yield

$$\begin{aligned}
E\left|\mathfrak{N}\left(\mathfrak{q}^{(\eta+1)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)\right|^2 &\leq \mathfrak{C}_2\left(E\left|\mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(\eta-1)}(\mathfrak{x})\right)\right|^2\right) \\
&\leq \cdots\leq \mathfrak{C}_2^{\eta-1}\left(E\left|\mathfrak{N}\left(\mathfrak{q}^{(2)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(1)}(\mathfrak{x})\right)\right|^2\right) \\
&\leq \mathfrak{C}_2^{\eta}\left(E\left|\mathfrak{N}\left(\mathfrak{q}^{(1)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(0)}(\mathfrak{x})\right)\right|^2\right) \qquad (3.4.9)
\end{aligned}$$

Again, from (3.4.5), applications of the inequality $|\sum_{i=1}^3 e_i|^2\leq 3\sum_{i=1}^3|e_i|^2$

$$\begin{aligned}
E\left|\mathfrak{N}\left(\mathfrak{q}^{(1)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(0)}(\mathfrak{x})\right)\right|^2 &\leq 3E\left|I_{\Delta,0}\ b\left(\mathfrak{x},\mathfrak{N}(\mathfrak{q}_0)\right)\right|^2 \\
&+3E\left|\Gamma(a+1)I^a_{\Delta,0}\ \sigma_1\left(\mathfrak{x},\mathfrak{N}(\mathfrak{q}_0)\right)\right|^2+3E\left|\int_0^{\mathfrak{x}}\sigma_2\left(\mathfrak{s},\mathfrak{N}(\mathfrak{q}_0)\right)\Delta B_{\mathfrak{s}}\right|^2.
\end{aligned}$$

Cauchy Schwartz inequality on the first two integral and Itô's Isometry on the third integral yields:

$$\begin{aligned}
E\left|\mathfrak{N}\left(\mathfrak{q}^{(1)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(0)}(\mathfrak{x})\right)\right|^2 \leq\ & 3\mathfrak{x}EI_{\Delta,0}\left|b\left(\mathfrak{x},\mathfrak{N}(\mathfrak{q}_0)\right)\right|^2 \\
&+3\left(\Gamma(a+1)\right)^2 h_a(\mathfrak{x},0)EI^a_{\Delta,0}\left|\sigma_1\left(\mathfrak{x},\mathfrak{N}(\mathfrak{q}_0)\right)\right|^2 \\
&+3EI_{\Delta,0}\left|\sigma_2\left(\mathfrak{x},\mathfrak{N}(\mathfrak{q}_0)\right)\right|^2
\end{aligned}$$

Linear growth condition yields:

$$\begin{aligned}
& E\left|\mathfrak{N}\left(\mathfrak{q}^{(1)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(0)}(\mathfrak{x})\right)\right|^2 \\
& \leq 3K^2\left(1+E|\mathfrak{N}(w_0)|^2\right)\left\{\mathfrak{x}^2+\mathfrak{x}+\left(\Gamma(a+1)\right)^2\left(h_a(\mathfrak{x},0)\right)^2\right\}.
\end{aligned}$$

Then,

$$\sup\left(E\left|\mathfrak{N}\left(\mathfrak{q}^{(1)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(0)}(\mathfrak{x})\right)\right|^2\right)$$

$$\leq 3K^2\left(1+E|\mathfrak{N}(w_0)|^2\right)\left\{\omega^2+\omega+(\Gamma(a+1))^2\left(h_a(\omega,0)\right)^2\right\} \quad (3.4.10)$$

As, $E\left|\mathfrak{N}\left(\mathfrak{q}^{(1)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(0)}(\mathfrak{x})\right)\right|^2$ is continuous, the application of (2.3.8), (2.3.10), (3.4.9) and (3.4.10) yield:

$$\begin{aligned} E\left|\mathfrak{N}\left(\mathfrak{q}^{(\eta+1)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)\right|^2 \leq\ & \mathfrak{C}_2^{\eta}\left(E\left|\mathfrak{N}\left(\mathfrak{q}^{(1)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(0)}(\mathfrak{x})\right)\right|^2\right) \\ & \leq 3K^2\left(1+E|\mathfrak{N}(w_0)|^2\right)\left\{\omega^2+\omega\right. \\ & \left.+(\Gamma(a+1))^2\left(h_a(\omega,0)\right)^2\right\}\frac{1}{\Gamma(\eta a+1)} \\ & \times\left[3\mathrm{L}^2\omega\left\{\mathfrak{x}+1+\omega^{a-1}(\Gamma(a+1))^2 h_a(\mathfrak{x},0)\right\}\right]^{\eta}. \end{aligned}$$

Therefore

$$\begin{aligned} & \sup_{\omega_0\leq\mathfrak{x}\leq\omega}\left(E\left|\mathfrak{N}\left(\mathfrak{q}^{(\eta+1)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)\right|^2\right) \\ & \leq \frac{M_0}{\Gamma(\eta a+1)}\left[3\mathrm{L}^2\omega\left\{\omega+1+\omega^{a-1}(\Gamma(a+1))^2 h_a(\omega,0)\right\}\right]^{\vartheta}, \end{aligned} \quad (3.4.11)$$

provided that

$$M_0:=3K^2\left(1+E|\mathfrak{N}(w_0)|^2\right)\left\{\omega^2+\omega+(\Gamma(a+1))^2\left(h_a(\omega,0)\right)^2\right\},$$

Thus, for any $\phi,\theta\in\mathbb{N}$ such that $\phi>\theta>0$, we have

$$\begin{aligned} & \left\|\mathfrak{N}\left(\mathfrak{q}^{(\phi)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(\theta)}(\mathfrak{x})\right)\right\|^2_{L^2_\Delta(\mathbb{P})} \\ & \leq \sum_{\eta=\theta}^{\phi}\left\|\mathfrak{N}\left(\mathfrak{q}^{(\eta+1)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)\right\|^2_{L^2_\Delta(\mathbb{P})} \\ & = \sum_{\eta=\theta}^{\phi}\int_0^{\omega} E\left|\mathfrak{N}\left(\mathfrak{q}^{(\eta+1)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)\right|^2\Delta\mathfrak{x} \\ & \leq M_0\sum_{\eta=\theta}^{\phi}\frac{(3\mathrm{L}^2\omega)^{\eta}}{(\eta+1)\Gamma(\eta a+1)}\left[\omega+1+\omega^{a-1}(\Gamma(a+1))^2 h_a(\omega,0)\right]^{\eta+1}\to 0, \end{aligned}$$

for sufficiently large ϕ,θ.

From Doob's maximal inequality for martingales, we get

$$\begin{aligned} & \sum_{\eta=1}^{\infty}\mathbb{P}\left[\sup_{\omega_0\leq\mathfrak{x}\leq\omega}\left|\mathfrak{N}\left(\mathfrak{q}^{(\eta+1)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)\right|>\frac{1}{\eta^2}\right] \\ & \leq M_0\sum_{\eta=1}^{\infty}\frac{\left[3\mathrm{L}^2\omega\left\{\omega+1+\omega^{a-1}(\Gamma(a+1))^2 h_a(\omega,0)\right\}\right]^{\eta}}{\Gamma(\eta a+1)}\,\eta^4<+\infty \end{aligned}$$

The Borel's cantelli Lemma yields:

$$\mathbb{P}\left\{\sup_{\omega_0\leq\mathfrak{x}\leq\omega}\left|\mathfrak{N}\left(\mathfrak{q}^{(\eta+1)}(\mathfrak{x})\right)-\mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)\right|>\frac{1}{\eta^2}\text{ for infinitely many }\ \eta\right\}=0,$$

so there exist a random variable $\mathfrak{N}\left(\mathfrak{q}(\mathfrak{x})\right)$ which is almost surely uniformly continuous on $[\omega_0, \omega]$, such that:

$$\mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right) = \mathfrak{N}\left(\mathfrak{q}^{(0)}(\mathfrak{x})\right) + \sum_{\theta=0}^{\eta-1}\left[\mathfrak{N}\left(\mathfrak{q}^{(\theta+1)}(\mathfrak{x})\right) - \mathfrak{N}\left(\mathfrak{q}^{(\theta)}(\mathfrak{x})\right)\right] \overset{\eta\to\infty}{\longrightarrow} \mathfrak{N}\left(\mathfrak{q}(\mathfrak{x})\right).$$

Since $\mathfrak{N}\left(\mathfrak{q}^{(\eta)}(\mathfrak{x})\right)$ is continuous in $\mathfrak{x}$ for any $\eta \in \mathbb{N}$, so $\mathfrak{N}\left(\mathfrak{q}(\mathfrak{x})\right)$ is also $\mathfrak{x}-$continuous. Therefore,

$$\begin{aligned}&\mathfrak{N}\left(w_0\right) + I_{\Delta,0}\ b\left(\mathfrak{x}, \mathfrak{N}(\mathfrak{q}^{(\eta)}(\mathfrak{x}))\right) + \Gamma(a+1) I^a_{\Delta,0}\ \sigma_1\left(\mathfrak{x}, \mathfrak{N}(\mathfrak{q}^{(\eta)}(\mathfrak{x}))\right)\\ &+ \int_0^{\mathfrak{t}} \sigma_2\left(\mathfrak{s}, \mathfrak{N}(\mathfrak{q}^{(\eta)}(\mathfrak{s}))\right) \Delta B_{\mathfrak{s}} \overset{\eta\to\infty}{\longrightarrow} \mathfrak{N}(\mathfrak{q}(\mathfrak{x})),\end{aligned}$$

for a stochastic process $\mathfrak{N}(\mathfrak{q}(\mathfrak{x}))$ satisfying (3.4.4). □

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